ENTROPY OF DIFFEOMORPHISMS OF SURFACES

by

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To my wife Beverley

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Summary.

We study the measure-theoretic and topological entropies of diffeomorphisms of surfaces. In the measure theoretic case we look for relations between Lyapunov exponents, Hausdorff dimension and the entropy of ergodic invariant measures.

First we describe the concept of measure-theoretic entropy in topological terms and discuss a general method of relating it with the Hausdorff dimension of ergodic invariant measures. This is done in a general setting, namely Lipschitz maps of compact metric spaces. The rest of the thesis is mainly directed to the study of diffeomorphisms of surfaces.

To apply a refinement of this general method to C^2 diffeomorphisms of surfaces we need Pesin's theory of non-uniform hyperbolicity, which we review in Chapter 2. Also in this chapter, we prove that the topological pressure of certain functions can be approximated by its restriction to the hyperbolic sets of the diffeomorphisms. This result is used in Chapter 3 to study the size of sets of generic points of ergodic measures supported on hyperbolic sets.

The main result of Chapter 3 is that if μ is an ergodic Borel f-invariant measure for a diffeomorphism $f:M \to M$ of a surface M. Then, provided the entropy $h_\mu(f) > 0$, the Hausdorff dimension of the set of generic points of μ is at least $1 + h_\mu(f)/\chi_\mu^+$, where χ_μ^+ is the positive Lyapunov exponent of μ.

In Chapter 4 we prove that if the family of local stable manifolds is Lipschitz, then for an ergodic measure μ,

$$h_\mu(f) = HD(\tilde{\mu}_x)\chi_\mu^+$$

for almost every $x \in M$. Here f is as in Chapter 3 and $\tilde{\mu}_x$ is a quotient measure defined by the family of local stable manifolds.

Finally, Chapter 5 is devoted to study the topological entropy of homoclinic closures by 'counting' homoclinic orbits.

Introduction.

The Ergodic Theory of diffeomorphisms studies the behaviour of the orbits of a given diffeomorphism $f:M \to M$ of a compact manifold M with respect to its invariant measures. The entropy of such a measure, [2], [28], [38], tells us how complicated the dynamic of the diffeomorphism is as a measure preserving transformation. Recent works, [16], [18], [39], have shown that, together with the asymptotic rates of expansion (Lyapunov exponents), the entropy also plays an important role in determining the size of an invariant measure. Since in many cases the sets on which such measures are concentrated have Lebesgue measure zero, the Hausdorff dimension is a natural concept to express how substantial the invariant measures are with respect to the metric structure of the manifold. In this Thesis we shall pursue the study of relations between entropy, Lyapunov exponents and Hausdorff dimension, with special emphasis on diffeomorphisms of surfaces.

The topological entropy, [1], is a topological analogue of the measure-theoretic entropy. Its connection with the growth rate of periodic points of diffeomorphisms has been studied by Bowen [4] and Katok [13]. These works imply a qualitative relation with the existence of transverse homoclinic points. We shall show that for C^2 diffeomorphisms of surfaces it is possible to quantify this relation in a way similar to the formulae for periodic points.

As each chapter has its own introduction we shall describe briefly the three very simple ideas on which this work is based.

We begin, Chapter 1, by characterizing the measure-theoretic entropy of continuous maps of compact metric spaces in topological terms. For this we shall discuss Bowen's definition of entropy for non-compact sets, [3], and Katok's definition of entropy by $(n,\varepsilon,\delta,)$-spanning sets, [3]. It turns out that using an idea of Misiurewicz [22] we obtain an elementary proof that these two definitions characterize the entropy of an ergodic measure. The first idea is Misiurewicz's use of the regularity of Borel measures on compact metric spaces. This will be used again, in Chapter 2, to prove that if we define the measure-theoretic pressure of continuous functions in terms of (n,ε,δ)-spanning sets, then by taking the supremum over all invariant measures we obtain the topological pressure as defined in [38].

Also in Chapter 1 we discuss a general method of obtaining upper bounds for the entropy of invariant ergodic measures for Lipschitz maps of compact metric spaces, in terms of the Hausdorff dimension of sets of full measure and the local Lipschitz constants of the maps, This second idea is mainly a combination of Manning's [18] and Ledrappier's [16]. Heuristically this can be described as follows: Let $T:X \to X$ be a Lipschitz map of a compact metric space X preserving an ergodic Borel probability measure μ. The entropy $h_\mu(T)$ measures (mod μ) how many different orbits T has, thus it should be at least the size

(Hausdorff dimension) of any set of measure 1 times the asymptotic rate of separation of the points in such a set, which we shall call the upper Lyapunov exponent. In order to prove this we use Bowen's characterization of entropy with its resemblance to the definition of Hausdorff dimension and also use Kingman's Subadditive Ergodic Theorem [14] to define a Lyapunov exponent for Lipschitz maps.

We said that the above is a general method because it will be used again in Chapter 3 in the case that $f:M \to M$ is a C^2 diffeomorphism of a surface M. Here we prove that if μ is an ergodic f-invariant Borel probability measure with $h_\mu(f) > 0$, then

$$HD(\{x \in M \mid \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) \to \int \phi d\mu \ , \ \forall \ \phi \ C(M)\}) \geq 1 + h_\mu(f)/X_\mu^+ \ ,$$

where $HD(\cdot)$ denotes the Hausdorff dimension, [2], X_μ^+ is the positive Lyapunov exponent, [26], and $C(M)$ is the set of all real valued continuous functions on M.

The third idea is to construct horseshoes, [23], so that dynamical quantities such as topological entropy and pressure of continuous functions can be approximated by the restriction to horseshoes. This is used in Chapter 2 to prove that the topological pressure of certain continuous functions of diffeomorphisms of surfaces can be obtained by taking the supremum over the restriction of the diffeomorphism and functions to horseshoes. This will be applied in Chapter 3 to study the function $\tau:\mathrm{Diff}^2(M) \to [0,1]$ on the space $\mathrm{Diff}^2(M)$ of C^2 diffeomorphisms $\mathrm{Diff}^2(M)$ of a surface M defined by

$$\tau(f) = \begin{cases} \sup\{h_\mu(f)/X_\mu^+ \mid \mu \text{ is ergodic with } h_\mu(f) > 0\} & \text{if } h(f) > 0 , \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove that $\tau(f) = \sup\{\tau(f|\Lambda) \mid \Lambda \text{ is hyperbolic}\}$ and as a corollary it follows that τ is lower semicontinuous. McCluskey and Manning [19] have shown that if Λ is a hyperbolic set then $\tau(f|\Lambda)$ is the Hausdorff dimension of the intersection of the unstable manifold of a point $x \in \Lambda$ with Λ.

This same third idea is used again in Chapter 5 to construct transverse homoclinic points, whose asymptotic rate will give us the topological entropy of homoclinic closures of diffeomorphisms of surfaces. For this we shall define an order for a homoclinic point that resembles the idea of period for periodic points.

Chapter 4 tries to generalize a theorem of Manning [18] and is based on Mañé's proof of Pesin's entropy formula, [20]. We prove that if the families of local stable manifolds of a C^2 diffeomorphisms $f:M \to M$ of a surface M are Lipschitz and μ is an ergodic measure f-invariant measure with $h_\mu(f) > 0$, then

$$h_\mu(f) = HD(\tilde{\mu}_x)X_\mu^+$$

for almost every $x \in M$. Here $\tilde{\mu}_x$ is the quotient measure on the local unstable manifold of x defined by the family of local stable manifolds and $HD(\tilde{\mu}_x)$ is the infimum of the Hausdorff dimension of sets of $\tilde{\mu}_x$ measure 1. Unfortunately we are not able to prove that the families of local stable manifolds are Lipschitz, neither do we know how to drop this condition.

A large part of this thesis is based on Pesin's theory of non-uniform hyperbolicity, [29], which we review in Chapter 2 following Katok [13].

The relationship between the different chapters may be represented by the following diagram:

$$\begin{tikzcd}
 & \text{Chapter 1} \arrow[d] & \\
\text{Chapter 2} \arrow[r] \arrow[d] & \text{Chapter 3} \arrow[r] & \text{Chapter 4} \\
\text{Chapter 5} & &
\end{tikzcd}$$

Our notation will be standard, for example $C\ell A$ denotes the closure of A. A compact smooth manifold M without boundary will be called simply a compact manifold, where we shall fix a smooth Riemannian metric, denoted by $d(\cdot,\cdot)$. The tangent space of M

at x will be denoted by T_xM and if $f:M \to M$ is a diffeomorphism $D_xf:T_xM \to T_{f(x)}M$ will denote the derivative of f at x .

All the measures considered will be Borel probability measures on compact metric spaces and hence regular (i.e. for every Borel set B and every $\varepsilon > 0$ there exists a closed set $C \subset B$ such that $\mu(B \setminus C) < \varepsilon$) .

The statements are numbered in order of appearance in each section of the corresponding chapter, for example Theorem 4.2.1 is the first statement of §2 of Chapter 4. Only the remarks that we shall refer to are numbered. We have chosen the most accessible and appropriate references for our use (thus sometimes ignoring the originals).

CHAPTER 1.

Entropy, Hausdorff Dimension and Lyapunov Exponents.

§0. Introduction.

In this chapter we shall describe the concept of measure theoretic entropy in topological terms, restricting our attention to the ergodic cases. This description of the entropy will be used in §3 to prove that: Let $T:X \to X$ be a Lipschitz map of a compact metric space X preserving an ergodic Borel probability measure μ. Then the entropy $h_\mu(T)$ of T with respect to μ satisfies

$$h_\mu(T) \leq HD(\mu)\chi_\mu^+ \quad ,$$

where $HD(\mu)$ equals the infimum of the Hausdorff dimension of sets of μ-measure 1 and χ_μ^+ is the upper Lyapunov exponent of T with respect to μ, which will be defined later on in this chapter.

In §1 we shall discuss two topological characterizations of measure theoretic entropy for continuous maps of compact metric spaces, introduced by Bowen [3] and Katok [13]. By going from one definition to the other and using Misiurewicz's ideas of the proof of the Variational Principle, [22], we avoid the combinatorial arguments of [3] and [13], obtaining an elementary way to prove that these two definitions characterize the entropy of ergodic measures.

The definition and some properties of Hausdorff dimension are discussed in §2. Finally, in §4, we apply the result of §3 to two examples of dynamical systems without differentiable structure, namely the shift space [10] and Smale's spaces [31].

Now we shall define entropy in the usual sense [2], [28], [37]. Let X be a compact metric space, $\mathcal{B}(X)$ the σ-algebra generated by the Borel sets of X and μ a probability measure on $(X,\mathcal{B}(X))$. Consider X as the probability space $(X,\mathcal{B}(X),\mu)$. A transformation $T:X \to X$ is <u>measure preserving</u> if T is measurable and $\mu(T^{-1}B) = \mu(B)$, $\forall\, B \in \mathcal{B}(X)$. We say that T is <u>ergodic</u> if $T^{-1}B = B$ implies that $\mu(B) = 0$ or $\mu(B) = 1$.

If ξ and $\mathcal{N}$ are two finite measurable partitions of X, let $\xi \vee \mathcal{N} = \{A_i \cap C_j \mid A_i \in \xi , C_j \in \mathcal{N}\}$ and write ξ_n for $\bigvee_{i=0}^{n-1} T^{-i}\xi$.

We define the <u>entropy of the partition</u> $\xi = \{A_1,\ldots,A_k\}$ as the number (Assume that $0\log 0 = 0$)

$$H_\mu(\xi) = -\sum_{i=1}^{k} \mu(A_i)\log\mu(A_i) ,$$

<u>the entropy of</u> T <u>with respect to</u> ξ as

$$h_\mu(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_n)$$

and <u>the entropy of</u> T <u>with respect to</u> μ as

$$h_\mu(T) = \sup\{h_\mu(T,\xi) \mid \xi \text{ is a finite partition of } X\} .$$

§1. Two characterizations of entropy in topological terms.

Let $T : X \to X$ be a continuous map of a compact metric space X. If A is a finite open cover of X and $E \subset X$, we write $E \prec A$ if E is contained in some element of A and $\{E_i\}_{i=1}^{\infty} \prec A$ if every $E_i \prec A$. Define $n_A(E)$ to be the largest non-negative integer such that $T^i E \prec A$ for every $0 \le i < n_A(E)$, $n_A(E) = 0$ if $E \not\prec A$ and $n_A(E) = \infty$ if $T^k E \prec A$ for all k's.

For $Y \subset X$, possibly non-compact, and $\lambda > 0$ define

$$m_{A,\lambda}(Y) = \lim_{\varepsilon \to 0} \inf\{\sum_i \exp - n_A(E_i)\lambda \mid \bigcup_i E_i \supset Y \text{ and } \exp - n_A(E_i) < \varepsilon\} ,$$

$$h_A(Y,T) = \inf\{\lambda \mid m_{A,\lambda}(Y) = 0\}$$

and

$$h(Y,T) = \sup_A h_A(Y,T) .$$

The above definition is due to Bowen [3], its connection with measure-theoretic entropy will be established in Theorem 1.1.1. Observe that for each cover A there exists a unique $\lambda_0 = h_A(Y,T)$ such that for $\lambda > \lambda_0$ $m_{A,\lambda}(Y) = 0$ and if $\lambda < \lambda_0$ $m_{A,\lambda}(Y) = \infty$.

Let d denote the metric on X and consider the metric $d_n(x,y) = \max_{0 \le i < n} d(T^i(x), T^i(y))$, which we shall call the d_n-<u>metric</u>.

We denote by $B_n(x,\varepsilon)$ the ε-ball centred at x in the d_n-metric.

Now suppose that T preserves an ergodic Borel probability measure μ . For $\varepsilon > 0$, $\delta > 0$ we say that a set $E \subset X$ is μ-(n,ε,δ)-spanning if $\mu(\bigcup_{x\in E} B_n(x,\varepsilon)) \geq 1-\delta$. Let $N(n,\varepsilon,\delta)$ denote the smallest cardinality of any μ-(n,ε,δ)-spanning set. Define

$$\underline{h}_\mu(T,\delta) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon,\delta)$$

and

$$\overline{h}_\mu(T,\delta) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon,\delta) .$$

The above definition is due to Katok [13].

Theorem 1.1.1.

Let $T:X \to X$ be a continuous map of a compact metric space X . If μ is an ergodic T-invariant Borel probability measure on X , then

$$h_\mu(T) = \inf_{\substack{Y\subset X\\ \mu(Y)=1}} h(Y,T) = \lim_{\delta\to 0} \underline{h}_\mu(T,\delta) = \lim_{\delta\to 0} \overline{h}_\mu(T,\delta) .$$

Proof.

The proof will be divided into three parts. First we prove that for any $\delta > 0$, $\overline{h}_\mu(T,\delta) \le h_\mu(T)$, following Katok [13]. Then, Part 2, using an idea from Misiurewicz [22] we simplify Bowen's proof of Theorem 1 of [3] to show that $h_\mu(T) \le \inf\{h(Y,T) \mid Y \subset X$ and $\mu(Y) = 1\}$. Finally, we prove that there exists $Y \subset X$ of full measure such that

$$h(Y,T) \le \lim_{\delta \to 0} \underline{h}_\mu(T,\delta) .$$

Part 1.

If ξ is a finite measurable partition of $(X,\mathcal{B}(X))$, then by the Shannon-McMillan-Brieman theorem [28] (S.M.B.)

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi_n(x)) = h_\mu(T,\xi) \ \mu\text{a.e.}$$

where $\xi_n(x)$ is the element of ξ_n containing x .

Let us choose a finite measurable partition ξ such that the diameter of each element of ξ is less than $\varepsilon/2$, for some $\varepsilon > 0$. Then $\xi_n(x) \subset B_n(x,\varepsilon)$. Let

$$A_{n,\varepsilon,\gamma} = \{x \in X : \mu(\xi_n(x)) > \exp{-n(h_\mu(T,\xi) + \gamma)}\} ,$$

since T is ergodic the S.M.B. theorem implies that $\mu(A_{n,\varepsilon,\gamma}) \to 1$ as $n \to \infty$ for any $\gamma > 0$. Fix $\delta > 0$ and choose n large enough so that $\mu(A_{n,\varepsilon,\gamma}) \ge 1-\delta$. The set $A_{n,\varepsilon,\gamma}$ contains at most exp $n(h_\mu(T,\xi)+\gamma)$ elements of ξ_n and can be covered by the same number of ε-balls in the d_n-metric, therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon,\delta) \le h_\mu(T,\xi) + \gamma .$$

Since γ is arbitrary and $h_\mu(T,\xi) \le h_\mu(T)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon,\delta) \le h_\mu(T) .$$

Part 2.

Let $N = \{A_1,\dots,A_k\}$ be a finite measurable partition of X . Choose $\varepsilon > 0$ such that $\varepsilon < \dfrac{1}{k\log k}$. Since μ is regular there exist compact sets $B_j \subset A_j$, $1 \le j \le k$, with $\mu(A_j \setminus B_j) < \varepsilon$. Let $\xi = \{B_0,B_1,\dots,B_k\}$ where $B_0 = X \setminus \bigcup_{j=1}^{k} B_j$. Since $\mu(B_0) \le k\varepsilon$ we have that

$$H_\mu(N|\xi) \le \mu(B_0)\log k \le k\varepsilon \log k < 1 ,$$

and therefore

$$h_\mu(T,N) \le h_\mu(T,\xi) + H_\mu(N|\xi) \le h_\mu(T,\xi) + 1 .$$

We summarize the above paragraph as follows.

Lemma 1.1.2.

Given a finite measurable partition N *there exists a partition* ξ *such that every* $x \in X$ *is in the closure of at most two elements of* ξ *and*

$$h_\mu(T,N) \le h_\mu(T,\xi) + 1 .$$

Note.

The above lemma is part of Misiurewicz's proof of variational principle, for more details see [37]page 189.

The proof of Part 2 will be completed by the next lemma that we borrow from Bowen [3].

Lemma 1.1.3.

Let ξ *be a finite measurable partition of* X *such that every* $x \in X$ *is in the closure of at most two elements of* ξ , *then for any* $Y \subset X$ *such that* $\mu(Y) = 1$,

$$h_\mu(T,\xi) \le h(Y,T) + \log 2 .$$

Proof.

By the S.M.B. theorem for μ-almost every $x \in X$ $\lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi_n(x))$ exists, furthermore

$$\int \lim -\frac{1}{n} \log \mu(\xi_n(x))d\mu = h_\mu(T,\xi) .$$

So by an application of Egorov's theorem for any $\alpha > 0$ there exists $N > 0$ such that

$$Y_N = \{x \in X \mid -\frac{1}{n} \log \mu(\xi_n(x)) \geq h_\mu(T,\xi)-\alpha \ , \ \ \forall\, n \geq N\}$$

has positive μ-measure, say $\mu(Y_N) > c > 0$. Notice that we are not using the ergodicity of μ .

Now let A be a finite open cover of X such that each element of it intersects at most two members of ξ . Suppose $\{E_i\}_{i=1}^{\infty}$ is a cover of Y such that $\exp -n_A(E_i) \leq \exp -N$ for each i . If $\beta \in \xi_{n_A(E_i)}$ intersects Y_N , then

$$\mu(\beta) \leq \exp(-n_A(E_i)(h_\mu(T,\xi)-\alpha)) \ .$$

Since $E_i \cap Y_N$ is covered by at most $2^{n_A(E_i)}$ of such β's ,

$$\mu(E_i \cap Y_N) \leq \exp(n_A(E_i)(\log 2 - h_\mu(T,\xi) + \alpha)) \ ,$$

so for $\lambda = h_\mu(T,\xi) - \log 2 - \alpha$

$$\sum_i \exp -\lambda n_A(E_i) \geq \sum_i (E_i \cap Y_N) \geq \mu(\cup E_i \cap Y_N) = \mu(Y_N) > c > 0$$

since $\mu(Y) = 1$, therefore $m_{A,\lambda}(Y) > c > 0$, which implies that

$$h_A(Y,T) \geq h_\mu(T,\xi) - \log 2 - \alpha \ ,$$

since α is arbitrary and $h_A(Y,T) \le h(Y,T)$ it follows that

$$h_\mu(T,\xi) \le h(Y,T) + \log 2 . \qquad \square$$

The last two lemmas imply that for any finite measurable partition $\mathcal{N}$ and any set $Y \subset X$ of measure 1 ,

$$h_\mu(T,\mathcal{N}) \le h(Y,T) + \log 2 + 1 .$$

Notice that we have not used the ergodicity of T , so it follows that for any $n > 0$ (replacing T by T^n)

$$h_\mu(T^n,\mathcal{N}) \le h(Y,T^n) + \log 2 + 1 .$$

Since $h(Y,T^n) = nh(Y,T)$ [3] $,Y,\mathcal{N}$ is arbitrary, Y is any set of measure 1 and by letting $n \to \infty$ it follows that

$$h_\mu(T) \le \inf\{h(Y,T) \mid Y \subset X , \mu(Y) = 1\} .$$

$\square$

<u>Part 3</u>.

Let $\{A_n\}_{n=1}^{\infty}$ be a collection of finite open covers such that diam $A_n \to 0$ as $n \to \infty$ and diam A_{n+1} < diam A_n . For each n let ε_n be a Lebesgue number for A_n , choose ε_{n+1} such that $\varepsilon_{n+1} < \varepsilon_n$.

Denote by $\underline{h}(T,\mu)$ the $\lim_{\delta\to 0} \underline{h}_\mu(T,\delta)$. Fix $\delta > 0$ and consider $E_n \subset X$ a μ-$(n,\varepsilon_n,\delta_n)$-spanning set of minimal cardinality, with $\delta_n = \delta/2^n$. Then for $\alpha > 0$ there exists a subsequence of n's, say $\{n_k\}$, such that

$$\frac{1}{n_k} \log N(n_k,\varepsilon_{n_k},\delta_{n_k}) \le \underline{h}(T,\mu) + \alpha .$$

Put $\tilde{E}_{n_k} = \bigcup_{x\in E_{n_k}} B_{n_k}(x,\varepsilon_{n_k})$ and let $\tilde{E} = \bigcap_{k=1}^{\infty} \tilde{E}_{n_k}$, clearly $\mu(E) \ge 1-\delta$ and for $n_k \ge m$

$$\sum_{x\in E_{n_k}} \exp-n_{A_m}(B_{n_k}(x,\varepsilon_{n_k}))(\underline{h}(T,\mu)+\alpha) \le$$

$$N(n_k,\varepsilon_k,\delta_{n_k})\exp-n_k(\underline{h}(T,\mu)+\alpha) .$$

But the last expression is less than 1, thus

$$h_{A_m}(E) \le \underline{h}(T,\mu) + \alpha$$

for large m's. Hence $h(E,T) \le \underline{h}(T,\mu)$, since α is arbitrary. And by letting $\delta \to 0$ we obtain a set $Y \subset X$ such that $\mu(Y) = 1$ and $h(Y,T) \le \underline{h}(T,\mu)$. □

Corollary 1.1.4.

If T *is a homeomorphism and* μ *is ergodic, then for any* $\delta > 0$

$$h_\mu(T) = \inf_{\substack{Y \\ \mu(Y)>0}} h(Y,T) = \underline{h}_\mu(T,\delta) = \overline{h}_\mu(T,\delta) .$$

Proof.

In Part 3 of the proof of Theorem 1.1.1 we proved that for any $1 > \delta > 0$ there exists $E \subset X$ such that $h(E,T) \le \underline{h}_\mu(T,\delta)$ and $\mu(E) \ge 1-\delta$. It is not difficult to verify that

$$h(\bigcup_{i=0}^{\infty} T^i E, T) = \sup_i h(T^i E, T) = h(E,T) .$$

Thus since μ is ergodic and T is a homeomorphism $\mu(\bigcup_{i=1}^{\infty} T^i E) = 1$.

Therefore, $h_\mu(T) = \inf\{h(Y,T) \mid Y \subset X , \mu(Y) > 0\}$, from which the corollary follows. $\square$

For a homeomorphism T and a finite open cover A of X, define $n_A^+(E) = n_A(E)$ and $n_A^-(E)$ to be the largest non-negative integer such that $T^{-i}E \prec A$ for $0 \le i < n_A^-(E)$. Let for $\lambda > 0$

$$\tilde{m}_{A,\lambda}(Y) = \lim_{\varepsilon \to 0} \inf\{\sum_i \exp-(n_A^+(E_i)+n_A^-(E_i))\lambda \mid \bigcup_i E_i \supset Y \text{ and } \exp-n_A^\pm(E_i)<\varepsilon \ \forall i\} .$$

Define

$$\tilde{h}_A(Y,T) = \inf\{\lambda \mid \tilde{m}_{A,\lambda}(Y) = 0\}$$

and

$$\tilde{h}\,(Y,T) = \sup_A \tilde{h}_A(Y,T) \ .$$

<u>Proposition 1.1.5.</u>

<u>If</u> T <u>is a homeomorphism, then for any</u> T-<u>invariant measure</u> μ <u>on</u> X <u>and</u> $Y \subset X$ <u>such that</u> $\mu(Y) = 1$

$$h_\mu(T) \le \tilde{h}\,(Y,T) \ .$$

<u>Proof.</u>

If T is a homeomorphism the S.M.B. theorem implies that

$$\int \lim_{n_1,n_2\to\infty} -\frac{1}{n_1+n_2} \log \mu(\xi_{n_1}^{n_2}(x))d\mu = h_\mu(T,\xi) \ ,$$

where $\xi_{n_1}^{n_2}(x) = \bigvee_{-n_1}^{n_2-1} T^{-i}\xi$. So we can modify Part 2 of the proof of Theorem 1.1.1 to prove that for any $Y \subset X$ of full measure

$$h_\mu(T) \le \tilde{h}\,(Y,T) + \log 2 \ .$$

But clearly $\tilde{h}\,(Y,T^n) = n\,\tilde{h}\,(Y,T)$, from where the proposition follows.

□

§2. Hausdorff dimension of a measure.

Let X be a metric space, for $Y \subset X$ then Hausdorff dimension $HD(Y)$ is defined by

$$HD(Y) = \inf\{\lambda \mid m_\lambda(Y) = 0\} ,$$

where $m_\lambda(Y) = \lim_{\varepsilon\to 0} \inf\{\sum_i (\operatorname{diam} E_i)^\lambda \mid \bigcup_i E_i \supset Y \text{ and } \operatorname{diam} E_i < \varepsilon , \forall i\}$.

We should note the way that Bowen's definition of entropy for non-compact sets resembles the definition of Hausdorff dimension.

In the following proposition we summarize some facts about Hausdorff dimension that we shall use later on, their proofs are standard and can be found in [2].

Proposition 1.2.1.

Let X be a compact metric space and $Y \subset X$, then:

i) there exists a unique $\lambda_0 \geq 0$ such that for $\lambda < \lambda_0$ $m_\lambda(Y) = \infty$ and for $\lambda_0 < \lambda$ $m_\lambda(Y) = 0$, and $\lambda_0 = HD(Y)$.

ii) If $\{Y_i\}$ is a countable collection of sets contained in X ,

$$HD(\bigcup_i Y_i) = \sup_i HD(Y_i) .$$

iii) If $T:X \to X$ is a Lipschitz map,

$$HD(TY) \leq HD(Y) . \qquad \square$$

Motivated by Theorem 1.1.1 and Ledrappier's definition of capacity of a measure [16], we introduce a notion of dimension of a measure.

If μ is a Borel probability measure on a compact metric space X, define the <u>Hausdorff dimension</u> of μ by

$$HD(\mu) = \inf\{HD(Y) \mid Y \subset X \text{ and } \mu(Y) = 1\} .$$

<u>Note</u>.

This definition has been introduced simultaneously by L.S. Young [39], although her 'entropy motivation' is based on local approaches to entropy.

<u>Remark 1.2.2</u>.

i) If ν is another Borel probability measure on X and $\mu \ll \nu$ then $HD(\mu) \leq HD(\nu)$.

ii) If $T:X \to X$ is a Lipschitz homeomorphism with Lipschitz inverse, i.e. a Lipeomorphism, of a compact metric space X, and μ is an ergodic T-invariant Borel probability measure on X, then

$$HD(\mu) = \inf\{HD(Y) \mid Y \subset X \text{ and } \mu(Y) > 0\} .$$

iii) One can prove that Ledrappier's capacity $C_L(\mu)$, [16], is greater or equal to $HD(\mu)$. In abstract spaces one could give examples where $HD(\mu) < C_L(\mu)$, but we do not know of any dynamical system where they are actually different. L.S. Young [39] has proved that if μ is an ergodic measure preserved by a C^2 diffeomorphism f of a surface, then if $h_\mu(f) > 0$, $C_L(\mu) = HD(\mu)$.

§3. An upper bound for the entropy of Lipschitz maps.

Let $T:X \to X$ be a Lipschitz map on a compact metric space X. Define $L(x) = \lim_{\varepsilon\to 0} \mathrm{Lip}(T|B(x,\varepsilon))$, where $\mathrm{Lip}(T|A)$ denotes the infimum of the Lipschitz constants of $T|A$. The function $L:X \to \mathbb{R}$ is well defined, furthermore it is measurable with respect to $\mathcal{B}(X)$ and in $L^1(X,\mathcal{B}(X),\mu)$ for any Borel probability measure μ.

Denote by $L_n(x) = \lim_{\varepsilon\to 0} \mathrm{Lip}(T^n|B(x,\varepsilon))$, then it is clear that

$$L_{m+n}(x) \le L_n(x)\, L_m(T^n(x)).$$

Then by the Subadditive Ergodic Theorem [14], there exists a T-invariant function $\chi:X \to \mathbb{R} \cup \{-\infty\}$ such that $\chi^+(x) = \max\{0,\chi(x)\} \in L^1(X,\mathcal{B}(X),\mu)$ for any T-invariant Borel probability measure μ,

$$\lim_{n\to\infty} \frac{1}{n} \log L_n(x) = \chi(x) \quad \mu \text{ a.e.}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int \log L_n(x)d\mu = \inf_n \frac{1}{n} \int \log L_n(x)d\mu = \int \chi(x)d\mu .$$

We shall call χ the upper Lyapunov exponent of T. If μ is ergodic, χ and χ^+ are constant almost everywhere. We shall denote these values by χ and χ^+ respectively.

Theorem 1.3.1.

Let $T:X \to X$ be a Lipschitz map of a compact metric space X and μ a T-invariant ergodic Borel probability measure on X. Then

$$h_\mu(T) \le HD(\mu)\chi_\mu^+ .$$

Note.

The above theorem generalizes Proposition 1 of Ledrappier [16] and Theorem 1 of Kushnirenko [15].

Proof.

If $A = \{A_1,\dots,A_t\}$ is an open cover of X, define

$$L_A(x) = \mathrm{Lip}(T \mid \bigcup_{i:x\in A_i} A_i) .$$

For $\{A_n\}_{n=1}^{\infty}$ a collection of finite open covers of X, define $A^n = A_n \vee A_{n-1} \vee .. \vee A_1$, clearly

$$L(x) \le L_{A^{n+1}}(x) \le L_{A^n}(x)$$

and L_{A^n} is a simple function for every $n \ge 1$, hence if $\mathrm{diam}\, A_n \to 0$

as $n \to \infty$,

$$\lim_{n\to\infty} L_{A^n}(x) = L(x)$$

and if μ is a Borel probability measure on X ,

$$\lim_{n\to\infty} \int \log^+ L_{A^n}(x)d\mu = \int \log^+ L(x)d\mu \quad ,$$

where we write ϕ^+ for $\max\{0,\phi\}$ for any $\phi : X \to \mathbb{R} \cup \{-\infty\}$.

Fix $n \geq 1$, for $m \geq n$ and $\varepsilon > 0$ consider the set

$$G^n_{\mu,m} = \{x \in X \mid \left|\frac{1}{k}\sum_{i=1}^{k-1} \log^+ L_{A^n}(T^i(x)) - \int \log^+ L_{A^n}(x)d\mu\right| \leq \varepsilon \text{ , } \forall\, k \geq m\} \quad ,$$

the Ergodic Theorem and an application of Egorov's theorem imply that $\mu(G^n_{\mu,m}) \uparrow 1$ as $m \to \infty$. For any $Y \subset X$, denote $G^n_{\mu,m} \cap Y$ by Y^n_m .

Let $\delta = HD(Y)$, with $Y \subset X$ and $\mu(Y) = 1$. Choose a cover $\mathcal{U}^n_m$ of Y^n_m by sets $U \subset Y^u_m$ such that $n_{A^n}(U) \geq m$ and $\sum_{U \in \mathcal{U}^n_m} (\operatorname{diam} U)^{\delta+\varepsilon} < 2^{-m}$.

Now let ℓ_n be a Lebesgue number for A^n , hence if $\operatorname{diam} T^i U \leq \ell_n$ for $0 \leq i < k$, $n_{A^n}(U) \geq k$. From the definition of L_{A^n} it follows that if $T^i U \prec A^n$ for $0 \leq i < k$, then for any $x \in U$

$$\operatorname{diam} T^k U \leq \operatorname{diam} U \prod_{i=0}^{k-1} L_{A^n}(T^i(x)) \text{ .}$$

Thus $n_{A^n}(U)$ satisfies $\ell_n \le \operatorname{diam} U \prod_{i=0}^{n_{A^n}(U)-1} L_{A^n}(T^i(x))$, $x \in U$.

Since $U \subset Y_m^n$ for $x \in U$ we have

$$\prod_{i=0}^{n_{A^n}(U)-1} L_{A^n}(T^i(x)) \le \exp n_{A^n}(U)\left(\int \log^+ L_{A^n}(x)d\mu + \varepsilon\right),$$

thus

$$\sum_{U\in\mathcal{U}_m^n} \exp-(\delta+\varepsilon)\left(\int \log^+ L_{A^n}(x)d\mu + \varepsilon\right)n_{A^n}(U) \le \ell_n^{-(\delta+\varepsilon)} \sum_{U\in\mathcal{U}_m^n} (\operatorname{diam} U)^{\delta+\varepsilon}$$

$$\le \ell_n^{-(\delta+\varepsilon)} 2^{-m} .$$

So combining all covers $\mathcal{U}_m^n$ for $m \ge n$, since ε is arbitrary, we obtain that

$$h_{A^n}(Y^n,T) \le HD(Y) \int \log^+ L_{A^n}(x)d\mu ,$$

where $Y^n = \bigcup_m Y_m^n$.

Set $\tilde{Y} = \bigcap_n Y^n$, then

$$h_{A^n}(\tilde{Y},T) \le HD(Y) \int \log^+ L_{A^n}(x)d\mu ,$$

hence by taking limits when $n \to \infty$ and Theorem 1.1.1 we have

$$h_\mu(T) \le HD(Y) \int \log^+ L(x)d\mu \quad .$$

Now we take the inf over all sets Y of measure 1 to obtain that

$$h_\mu(T) \le HD(\mu) \int \log^+ L(x)d\mu \quad .$$

The proof of the theorem will be completed if we prove that $h_\mu(T^n) \le HD(\mu) \int \log^+ L_n(x)d\mu$, for all n .

To apply the above procedure to T^n we decompose μ , if necessary, into its T^n ergodic components. The ergodicity of μ implies that there exists a finite number of sets $\Lambda_i \subset X$, say $\{\Lambda_1,\ldots,\Lambda_r\}$, such that $T^{-1}\Lambda_i = \Lambda_{i+1}$ and $T^{-1}\Lambda_r = \Lambda_1$, $\mu(\Lambda_i) = \mu(\Lambda_{i+1})$ and T^n is ergodic with respect to μ_i the conditional measure on Λ_i , given by $\mu_i(B) = \dfrac{\mu(B\cap\Lambda_i)}{\mu(\Lambda_i)}$. Therefore

$$h_{\mu_i}(T^n) \le HD(\mu_i) \int \log^+ L_n(x)d\mu_i \ .$$

Since T is Lipschitz, $HD(\mu_i) = HD(\mu)$ and by the Ergodic Decomposition Theorem of Entropy [10],

$$h_\mu(T^n) \le HD(\mu) \int \log^+ L_n(x)d\mu \quad ,$$

and since $h_\mu(T^n) = nh_\mu(T)$, the result follows. □

The <u>topological entropy</u> $h(T)$ of a continuous map $T:X \to X$, X as above, can be defined as $h(T) = h(X,T)$, see [1] and [3]. The Variational Principle [37] says that $h(T) = \sup_{\mu} h_{\mu}(T)$. So if $\Lambda \subset X$ is such that $\mu(\Lambda) = 1$ for all ergodic T-invariant measures μ, then $h(T) \le HD(\Lambda)\log^{+}k$, where k is a Lipschitz constant for T.

The <u>non-wandering set</u> $\Omega(T) = \Omega_1(T)$ of T is defined as the set

$$\{x \in X \mid \text{for every neighbourhood } U \text{ of } x \ \exists\ n \ge 1 \text{ with } T^{-n}U \cap U \neq \phi\} .$$

Define $\Omega_2(T) = \Omega(T|\Omega_1(T)),\ldots,\Omega_n(T) = \Omega(T|\Omega_{n-1}(T))$, the intersection $\bigcap_{n=1}^{\infty} \Omega_n(T)$ is called <u>the centre of</u> T and is denoted by $\Omega_{\infty}(T)$. It is well known, for instance see [37], that $\mu(\Omega_{\infty}(T)) = 1$ for all T-invariant measures μ.

<u>Corollary 1.3.2</u>.

<u>For</u> $T:X \to X$ <u>as above</u>

$$h(T) \le HD(\Omega_{\infty}(T)) \log^{+}k ,$$

<u>where</u> $k = \mathrm{Lip}(T|\Omega_{\infty}(T))$. □

<u>Note</u>.

This corollary generalizes results of Bowen [7] and Kushnirenko [15].

If $T:X \to X$ is a lipeomorphism we can define χ for T^{-1}. So in order to avoid any confusion we shall denote by $\chi(x,T)$ the upper Lyapunov exponent for T at x and respectively $\chi(x,T^{-1})$ for T^{-1}.

Corollary 1.3.3.

Let $T:X \to X$ *be a lipeomorphism of a compact metric space* X *preserving an ergodic Borel probability measure* μ. *If* $h_\mu(T) > 0$, *then*

$$h_\mu(T)\left(\frac{1}{\chi_\mu^+(T)} + \frac{1}{\chi_\mu^+(T^{-1})}\right) \le HD(\mu) ,$$

where $\chi_\mu^+(T)$ *and* $\chi_\mu^+(T^{-1})$ *denote the upper Lyapunov exponents for* T *and* T^{-1} *respectively.* □

Corollary 1.3.3 follows from similar arguments to the ones used in the proof of Theorem 1.3.1 and Corollary 1.1.4. We shall sketch the proof for diffeomorphisms.

Corollary 1.3.4.

Let $f:M \to M$ *be a* C^1 *map of a compact manifold* M *preserving an ergodic Borel probability measure* μ. *Then if* $h_\mu(f) > 0$:

i) $h_\mu(f) \le HD(\mu)\chi_\mu^+$, *where* χ_μ^+ *equals the largest positive Lyapunov exponent.*

ii) $h(f) \le HD(\Omega_\infty(f)) \sup_{x \in \Omega_\infty(f)} \log||D_x f||$.

iii) <u>If</u> f <u>is a diffeomorphism</u>, <u>then if</u> $h_\mu(f) > 0$

$$h_\mu(f)\left(\frac{1}{\chi_r^\mu} - \frac{1}{\chi_1^\mu}\right) \le HD(\mu)$$

<u>where</u> $\chi_1^\mu < \chi_2^\mu < .. < 0 < ... < \chi_r^\mu$ <u>are the Lyapunov exponents</u> <u>of</u> f <u>with respect to</u> μ .

The Lyapunov exponents for diffeomorphisms will be defined in the next chapter, and we refer to [26].

<u>Note</u>.

L.S. Young [39] has proved that if f is C^2 and M a surface with $h_\mu(f) > 0$, then $h_\mu(f)\left(\frac{1}{\chi_2^\mu} - \frac{1}{\chi_1^\mu}\right) = HD(\mu)$.

<u>Proof</u>.

Let $D_x f$ denote the derivative of f at x . Observe that $L(x) = ||D_x f||$, so i) and ii) are trivial. We shall prove that if f is a diffeomorphism and $h_\mu(f) > 0$ then

$$h_\mu(f) \le HD(\mu) \frac{\int \log||D_x f||d\mu \int \log||D_x f^{-1}||d\mu}{\int \log||D_x f||d\mu + \int \log||D_x f^{-1}||d\mu} .$$

Then we proceed as in Theorem 1.3.1 to consider f^n , for $n > 0$.

For $\alpha > 0$ choose a finite open cover A of M such that $||D_x f||$ and $||D_x f^{-1}||$ do not change more than α in each element of A. For $\varepsilon > 0$ consider the set

$$G_{\mu,n_1,n_2} = \{x \in M | \quad \left|\frac{1}{m}\sum_{k=0}^{m-1} \log(||D_{f^k x} f|| + \alpha) - \int \log(||D_x f|| + \alpha) d\mu\right| \le \varepsilon$$

$$\text{and} \qquad \left|\frac{1}{m}\sum_{k=0}^{m-1} \log(||D_{f^{-k} x} f^{-1}|| + \alpha) - \int \log(||D_x f^{-1}|| + \alpha) d\mu\right| \le \varepsilon$$

$$\forall\, m \ge n_1, n_2\} \ ,$$

the Ergodic Theorem implies that $\mu(G_\mu, n_1, n_2) \uparrow 1$ as $n_1, n_2 \to \infty$. Using an appropriate cover of G, n_1, n_2, Corollary 1.1.4 and the Mean Value Theorem we obtain that for U an element of such cover

$$\exp - \left(\int \log(||D_x f|| + \alpha) d\mu + \varepsilon\right) n^+(U) \le \ell^{-1} \text{ diam } U$$

and

$$\exp - \left(\int \log(||D_x f^{-1}|| + \alpha) d\mu + \varepsilon\right) n^-(U) \le \ell^{-1} \text{ diam } U \ .$$

Write $\phi^u = \int \log(||D_x f|| + \alpha) d_\mu + \varepsilon$ and $\phi^s = \int \log(||D_x f^{-1}|| + \alpha) d_\mu + \varepsilon$,

so taking sum over all U's

$$\sum_{U's} \exp -(n^+(U) + n^-(U))(\delta+\varepsilon) \frac{\phi^u \phi^s}{\phi^u + \phi^s} =$$

$$\sum_{U's} (\exp -n+(U)\phi^u)^{(\delta+\varepsilon)\frac{\phi^s}{\phi^u+\phi^s}} (\exp -n^-(U)\phi^s)^{(\delta+\varepsilon)\frac{\phi^u}{\phi^u+\phi^s}}$$

$$\leq \sum_{U's} (\ell^{-1}\mathrm{diam}U)^{(\delta+\varepsilon)\frac{\phi^s}{\phi^u+\phi^s}} (\ell^{-1}\mathrm{diam}\ U)^{(\delta+\varepsilon)\frac{\phi^u}{\phi^u+\phi^s}}$$

$$= \sum_{U's} (\ell^{-1}\mathrm{diam}\ U)^{(\delta+\varepsilon)\left(\frac{\phi^s}{\phi^u+\phi^s}+\frac{\phi^u}{\phi^u+\phi^s}\right)} = \sum_{U's} (\ell^{-1}\mathrm{diam}\ U)^{\delta+\varepsilon} ,$$

where $\delta = HD(\bigcup_{n_1,n_2} G_{\mu,n_1,n_2})$, then it follows that

$$h_\mu(f) \leq \delta \frac{\int \log||D_xf||d\mu \int \log||D_xf^{-1}||d\mu}{\int \log||D_xf||d\mu + \int \log||D_xf^{-1}||d\mu} .$$

Since $\bigcup_{n_1,n_2} G_{\mu,n_1,n_2}$ can be replaced by $\bigcup_{n_1,n_2} G_{\mu,n_1,n_2} \cap Y$ where Y is any set of measure 1 , we can prove that

$$h_\mu(f) \leq HD(\mu) \frac{\int \log||D_xf||d\mu \int \log||D_xf^{-1}||d\mu}{\int \log||D_xf||d\mu + \int \log||D_xf^{-1}||d\mu} .$$

□

§4. Examples.

i) Subshifts of finite type.

Let $\sigma:\Sigma_N \to \Sigma_N$ denote the full two-sided shift on N symbols, see [10] for more details. Given $a > 1$ we define a metric on Σ_N by $d(x,y) = a^{-k}$ where k is the largest non-negative integer for which $x_i = y_i$ for all $0 \le |i| < k$. It is clear that σ is Lipschitz in this metric and $\chi(x,\sigma) = \chi(x,\sigma^{-1}) = \log a$. For any σ-invariant ergodic measure μ Corollary 1.3.3 implies that $h_\mu(\sigma) \le HD(\mu) \dfrac{\log a}{2}$, it is not difficult to show that equality holds. Therefore if μ_1 and μ_2 are two ergodic σ-invariant measures on Σ_N then

$$\frac{h_{\mu_1}(\sigma)}{h_{\mu_2}(\sigma)} = \frac{HD(\mu_1)}{HD(\mu_2)} ,$$

which says that the ratio of the dimensions of the measures μ_1 and μ_2 depends only on the ratio of their entropies.

Similarly, if $\sigma_A:\Sigma_A \to \Sigma_A$, $\sigma_B:\Sigma_B \to \Sigma_B$ are two subshifts of finite type contained in Σ_N, then $HD(\Sigma_A)/HD(\Sigma_B) = h(\sigma_A)/h(\sigma_B)$ and if σ_A is topological conjugate to σ_B then $HD(\Sigma_A) = HD(\Sigma_B)$.

ii) Smale. spaces.

A Smale space is a compact metric space Ω with a homeomorphism T that admits local coordinates, see [31] for precise definitions, generalizing the notion of basic sets for Axiom A diffeomorphisms [35]. D. Fried [11] has proved that there exists a metric d in Ω such that T is a Lipeomorphism, therefore by Corollary 1.3.3 if $h(T) > 0$ then $HD(\Omega) > 0$.

CHAPTER 2.

Lyapunov Exponents and Pressure of Diffeomorphisms of Surfaces.

§0. Introduction.

In this chapter we summarize some results of the theory of non-uniform hyperbolic sets, which we shall use in this thesis. The basic results are due to Pesin [29], [30] and Katok [13]. Actually we shall follow [13], especially sections 2 and 3.

Our aim is to extend methods of Katok and Newhouse to prove that for a C^2 diffeomorphism $f:M \to M$ of a surface M and $\phi:M \to \mathbb{R}$ a continuous map, if

$$P(f,\phi) = \sup_{\mu}\{h_\mu(f) + \int \phi d\mu\} = \sup_{\substack{\mu \text{ nonzero} \\ \text{exponents}}} \{h_\mu(f) + \int \phi d\mu\} ,$$

then

$$P(f,\phi) = \sup_{\Lambda \text{ uniformily hyperbolic}} P(f|\Lambda,\phi|\Lambda) .$$

Moreover, the sup can be taken over zero topological dimensional sets.

The proof of this result is based on a talk given by S. Newhouse at I.H.E.S. (1981) for entropy, $\phi \equiv 0$. In that case the result is attributed to Katok [13] and here we have just modified a few arguments and definitions to extend the result to the pressure of continuous functions [38]. We shall need an anologue of Katok's definition of entropy, see Chapter 1, for pressure of continuous function relative to an invariant measure μ . We shall denote this concept by $P_\mu(f,\phi)$, and we prove a variational principle

$$P(f,\phi) = \sup_\mu P_\mu(f,\phi) .$$

§1. Non-uniform hyperbolicity.

Let $f:M \to M$ be a C^1 diffeomorphism of a compact C^∞ manifold M . Fix a C^∞ Riemannian metric on M . A closed f-invariant set $\Lambda \subset M$ is called <u>uniformly hyperbolic</u> or simply <u>hyperbolic</u> if there exist a splitting $T_xM = E_x^s \oplus E_x^u$ for each $x \in \Lambda$, which varies continuously with $x \in \Lambda$, constants $\lambda > 1$ and $c > 0$ such that for $n \geq 0$

i) $D_xfE_x^s = E_{fx}^s$, $D_xfE_x^u = E_{fx}^u$,

ii) $||D_xf^nv|| \leq c\lambda^{-n}||v||$ for $v \in E_x^s$

$||D_xf^nv|| \geq c^{-1}\lambda^n||v||$ for $v \in E_x^u$.

A diffeomorphism $f:M \to M$ is said to satisfy <u>Axiom A</u> if $\Omega(f)$ is hyperbolic and $Per(f) = \{x|f^nx = x \text{ some } n > 0\}$ is dense in $\Omega(f)$. The dynamics of these diffeomorphisms are fairly well understood, see [6], [8], [35] .

In [29] Pesin developed a theory of non-uniform hyperbolicity to study ergodic properties of diffeomorphisms that preserve measures absolutely continuous with respect to the Lebesgue measure of the manifold; it has been observed [13], [32] that a good deal of Pesin's theory is still true without any assumption on the invariant measures. The name of non-uniform hyperbolicity will be evident after the following definitions.

Now let $f:M \to M$ be a C^2 diffeomorphism of a compact C^∞ finite dimensional manifold M. As above fix a C^∞ Riemannian metric on M. For $\chi > 0$, $\ell > 1$ denote by $\Lambda_{\chi,\ell}$ the set of all points $x \in M$ with the following properties: There exists a splitting $T_xM = E_x^s \oplus E_x^u$ of the tangent space at x such that for any $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$ for $v \in D_xf^mE_x^s$

$$||D_{f^mx}f^nv|| \le \ell \exp{-n\chi} \exp(\chi 10^{-3}(|m|+n))||v||$$

$$||D_{f^mx}f_v^{-n}|| \ge \ell^{-1} \exp{n\chi} \exp(-\chi 10^{-3}(|m|+n))||v|| ,$$

for $v \in D_xf^mE_x^u$

$$||D_{f^mx}f^nv|| \ge \ell^{-1} \exp{n\chi} \exp(-\chi 10^{-3}(|m|+n))||v||$$

$$||D_{f^mx}f_v^{-n}|| \le \ell \exp{-n\chi} \exp(\chi 10^{-3}(|m|+n))||v||$$

and the angle $\gamma(x)$ between E_x^s and E_x^u satisfies

$$\gamma(f^m x) \geq \ell^{-1} \exp -\chi 10^{-3}|m| \quad .$$

For a non-negative integer $k \leq \dim M$ denote by

$$\Lambda_{\chi,\ell}^k = \{x \in \Lambda_{\chi,\ell} \mid \dim E_x^s = k\} \quad .$$

Obviously if $\chi_1 \leq \chi_2$, $\ell_1 \geq \ell_2$ then $\Lambda_{\chi_1,\ell_1}^k \supset \Lambda_{\chi_2,\ell_2}^k$.

<u>Proposition 2.1.1</u>. [29]

i) <u>The sets</u> $\Lambda_{\chi,\ell}^k$ <u>are closed</u>.

ii) <u>The subspaces</u> E_x^s , E_x^u <u>depend on</u> x <u>continuously on the set</u> $\Lambda_{\chi,\ell}^k$.

iii) <u>For every integer</u> q <u>and</u> $\ell > 1$ <u>there exists</u> $L = L(\chi,q,\ell)$ <u>such that</u> $f^q(\Lambda_{\chi,\ell}^k) \subset \Lambda_{\chi,L}^k$. □

An f-invariant set $\Lambda \subset M$ is said to be <u>non-uniformily hyperbolic</u> if there exists $\chi > 0$ such that $\Lambda = \bigcup_{\ell > 1} \Lambda_{\chi,\ell}$. The sets $\Lambda_{\chi,\ell}^k$ will be called <u>Pesin's sets</u>.

For $v \in T_x M$ the number

$$\chi^+(x,v) = \limsup_{n\to\infty} \frac{1}{n} \log||D_x f^n||$$

is called the <u>upper Lyapunov exponent</u>. The function $\chi^+(x,\cdot)$ takes at most $N = \dim M$ values and generates a filtration of T_xM by subspaces

$$\{0\} = L_0(x) \subset L_1(x) \subset L_2(x) \subset .. \subset L_{r(x)}(x) = T_xM ,$$

namely there are numbers $\chi_1(x) < \chi_2(x) < .. < \chi_{r(x)}(x)$ such that $\chi^+(x,v) = \chi_i(x)$ for $v \in L_i(x) \setminus L_{i-1}(x)$, we shall call these numbers the <u>Lyapunov exponents</u> of f at x . The number $k_i(x) = \dim L_i(x) - \dim L_{i-1}(x)$ is called the multiplicity of the i-th exponent.

Under certain conditions on x called regularity, see [26], [29], it can be proved that $\lim\limits_{n\to\infty} \frac{1}{n} \log||D_x f^n v||$ exists $\forall\, v \neq 0$. Osedelec's Multiplicative Ergodic Theorem [26] implies that the set of regular points has measure 1 for any f-invariant measure μ . The functions $\chi_i(x)$, $k_i(x)$, $r(x)$ are f-invariant and measurable with respect to any Borel f-invariant measure μ . Therefore, if μ is ergodic, these functions are constant almost everywhere and we shall denote them by χ_i^μ, k_i^μ, r^μ respectively. We shall call the numbers $\chi_1^\mu < .. < \chi_{r^\mu}^\mu$ the <u>Lyapunov exponents of</u> μ or simply <u>the exponents of</u> μ .

<u>Proposition 2.1.2</u>. [26]

<u>Let</u> x <u>be a regular point for</u> f <u>with Lyapunov exponents</u> $\chi_1(x),\dots,\chi_{r(x)}(x)$ <u>different from zero</u>, $\chi(x) = \min\limits_{1\le i\le r(x)} |\chi_i(x)|$ <u>and</u>

$k(x) = \sum_{i:\chi_i(x)<0} k_i(x)$ be the number of negative exponents with their multiplicities. Then $x \in \Lambda^{k(x)}_{\chi(x),\ell}$ for some $\ell > 1$. □

This proposition and the Multiplicative Ergodic Theorem imply the following statement.

Corollary 2.1.3. [13]

For any Borel probability f-invariant measure μ with nonzero exponents, $\mu(\Lambda) = 1$, where Λ denotes $\bigcup_{\chi,\ell} \Lambda_{\chi,\ell}$. Moreover if μ is ergodic, then $\mu(\bigcup_{\ell>1} \Lambda^k_{\chi,\ell}) = 1$, where $\chi = \min_i |\chi^\mu_i|$ and $k = \sum_{i:\chi^\mu_i<0} k^\mu_i$. □

Now we are going to define the Lyapunov metric near a regular point. This will allow us to consider the linear parts of f along the orbit of such a point as hyperbolic operators.

Proposition 2.1.4. [29]

There exists a number $\varepsilon_0 > 0$ which depends only on f such that for every point $x \in \Lambda^k_\chi = \bigcup_{\ell>1} \Lambda^k_{\chi,\ell}$ we can find a neighbourhood $B(x)$ and a diffeomorphism $\Phi_x : B^k_{\varepsilon_0} \times B^{N-k}_{\varepsilon_0} \to B(x)$ (B^i_ε - Euclidean ε-ball around the origen in $\mathbb{R}^i$) with the following properties:

i) The image of the standard Euclidean metric in $B^k_{\varepsilon_0} \times B^{N-k}_{\varepsilon_0}$ is a Riemannian metric $\langle\cdot,\cdot\rangle'_x$ on $B(x)$ which generates the norm $\|\ \|'_x$

in each tangent space T_yM, $y \in B(x)$, and is connected with the norm $\| \ \|$, generated by the given Riemannian metric, by the following inequalities:

$$K_1 \le \frac{\| \ \|'_x}{\| \ \|} \le K_2 A(x)$$

where K_1, K_2 are constants and $A(x)$ is a Borel function of x such that given any integer m :

$$A(f^m(x)) \le A(x) \min\{(3/2)^{|m|}, \exp 2\times10^{-3}|m|\}$$

and

$$\sup_{x \in \Lambda^k_{\chi,\ell}} A(x) = A^k_{\chi,\ell} < \infty .$$

ii) The map

$$f_x = \Phi^{-1}_{f(x)} \circ f \circ \Phi_x : B^k_{\varepsilon_0} \times B^{N-k}_{\varepsilon_0} \to \mathbb{R}^N$$

has the form

$$f_x(u,v) = (A_x u + h_{1x}(u,v), B_x v + h_{2x}(u,v))$$

where $h_{1x}(0,0) = h_{2x}(0,0) = 0$, $Dh_{1x}(0,0) = Dh_{2x}(0,0) = 0$ and

$$||A_x|| < \exp - \frac{99}{100} X$$

$$||B_x^{-1}||^{-1} < \exp - \frac{99}{100} X \quad .$$

(All norms are Euclidean here and below, i.e. this section.)

<u>Set</u> $\lambda(X) = \max\{\frac{1}{2}, \exp - \frac{99}{100} X\}$, <u>then</u>, <u>for</u> $z = (u,v)$, $h_x(z) = (h_{1x}(z), h_{2x}(z))$:

$$||D_{z_1} h_x - D_{z_2} h_x|| \le K\lambda(X)||z_1 - z_2||$$

<u>with</u> K <u>an absolute constant</u>.

iii) <u>The metric</u> $\langle\cdot,\cdot\rangle_x'$ <u>depends on</u> x <u>continuously on any set</u> $\Lambda_{X,\ell}^k$.

iv) <u>For any</u> $z \in M$ <u>the decomposition</u>

$$T_z M = D\Phi_x \mathbb{R}^k \oplus D\Phi_x \mathbb{R}^{N-k}$$

<u>depends continuously on</u> x <u>for such</u> $x \in \Lambda_{X,\ell}^k$ <u>that</u> $z \in B(x)$. □

These last two statements are due to Katok [13].

For $x \in \Lambda_{X,\ell}^k$ let $C(x) = \Phi_x(B_{\varepsilon(x)}^k \times B_{\varepsilon(x)}^{N-k})$ where

$$\varepsilon(x) = \frac{(1 - \lambda(X))^2}{100} (2K)^{-1}(A(x))^{-1} ,$$

we shall call the new neighbourhood $C(x)$ the <u>standard x-box</u>. It follows from Proposition 2.1.4(i) that

$$\varepsilon(f^m(x)) \geq \varepsilon(x) \max\{(3/2)^{-|m|}, \exp -2\times 10^{-3}|m|)\}$$

and

$$\varepsilon(x) \geqslant \frac{(1-\lambda(X))^2}{100}(2K)^{-1}(A^k_{X,\ell})^{-1} = \varepsilon(k,X,\ell) > 0 .$$

Now fix $h \in (0,1]$ and for $x \in \Lambda^k_{X,\ell}$ set

$$C(x,h) = \Phi_x(B^k_{h\varepsilon(x)} \times B^{N-k}_{h\varepsilon(x)}) .$$

Let us denote by $\mathcal{U}^{\gamma,\delta,h}_x$, with $0 < \gamma < 1$ and $\delta > 0$, the following set of $N-k$ dimensional submanifolds of $C(x,h)$:

$$\mathcal{U}^{\gamma,\delta,h}_x = \{\Phi_x(\text{graph}\phi):\phi \in C^1(B^{N-k}_{h\varepsilon(x)},B^k_{h\varepsilon(x)}),||\phi(0)|| \leq \delta, ||D\phi|| \leq \gamma\} .$$

Obviously if $\gamma_1, \gamma_2, \delta_1 \geq \delta_2$ then $\mathcal{U}^{\gamma_1,\delta_1,h}_x \supset \mathcal{U}^{\gamma_2,\delta_2,h}_x$. We define in a similar way the set $\mathcal{S}^{\gamma,\delta,h}_x$ of k-dimensional submanifolds of $C(x,h)$:

$$\mathcal{S}^{\gamma,\delta,h}_x = \{\Phi_x(\text{graph}\phi):\phi \in C^1(B^k_{h\varepsilon(x)},B^{N-k}_{h\varepsilon(x)}),||\phi(0)|| \leq \delta,||D\phi|| < \gamma\} .$$

For $\chi > 0$, let $\gamma(\chi) = \dfrac{1 - \lambda(\chi)}{20}$.

Proposition 2.1.5. [13]

Suppose that $x \in \Lambda^k_{\chi,\ell}$, $\delta \le \dfrac{h\varepsilon(x)}{2}$ *and* $B \in \mathcal{U}_x^{\gamma(\chi),\delta,h}$.

Then

i) $fB \cap C(f(x)) \in \mathcal{U}_{f(x)}^{\lambda(\chi)\gamma(\chi),\delta(\frac{1+\lambda(\chi)}{2}),h}$;

ii) *for any* $y_1, y_2 \in B$

$$d'_{f(x)}(f(y_1), f(y_2)) > (\tfrac{1}{2} + \frac{1}{2\lambda(\chi)})d'_x(y_1, y_2) ,$$

where $d'_x(\cdot,\cdot)$ *is the distance function generated by the metric* $\langle\cdot,\cdot\rangle'_x$. □

Let us denote the neighbourhood

$$\Phi_x(B^k_{h\varepsilon(k,\chi,\ell)/2} \times B^{N-k}_{h\varepsilon(k,\chi,\ell)/2})$$

of a point $x \in \Lambda^k_{\chi,\ell}$ by $C(x,h,k,\chi,\ell)$. Sometimes for convenience of notation (if k,χ,ℓ are fixed) we shall write ε instead of $\varepsilon(k,\chi,\ell)$ and $\hat{C}(x,h)$ instead of $C(x,h,k,\chi,\ell)$.

Furthermore, we shall call any manifold of the form $B \cap \hat{C}(x,h)$ where $B \in \mathcal{U}_x^{\gamma(\chi),h\varepsilon/4,h}$ *an admissible* (u,h)-*manifold near* x and any manifold of the form $B \cap \hat{C}(x,h)$ where $B \in S_x^{\gamma(X),h\varepsilon/4,h}$ an admissible (s,h)-manifold near x.

Corollary 2.1.6. [13]

For any k, $X > 0$, $\ell > 0$, $\beta < 1/4$, $0 < h \leq 1$ there exists a number $\kappa = \kappa(k,X,\ell,\beta,h)$ such that if $x,y \in \Lambda_{X,\ell}^k$, $d(x,y) < \kappa$, $B \in \mathcal{U}_y^{A\beta\gamma(X),h\beta\varepsilon,h}$ then $B \cap C(x,h,k,X,\ell)$ is an admissible (u,h)-manifold near x. □

The following proposition is a weak version of Katok's Main Lemma, its proof is contained in pages 158-160 of [13]. A similar statement is true for admissible (s,h)-manifolds.

Proposition 2.1.7. [13]

There exists $\psi = \psi(k,X,h,\ell)$ such that for given $x,y \in \Lambda_{X,\ell}^k$ with $f^n(x) \in \Lambda_{X,\ell}^k$ and $d(y,f^n(x)) < \psi$, for some $n \geq 0$, if B_0 is an admissible (u,h)-manifold near x, then B_1 defines as follows

$$B_0^1 = fB_0$$
$$B_0^i = f(B_0^{i-1} \cap C(f^{i-1}(x),h)) \quad i = 2,\ldots,n-1$$
$$B_1 = fB_0^{n-1} \cap \hat{C}(y,h)$$

is an admissible (u,h)-manifold near y. □

Now we shall restrict our attention to the case $\dim M = 2$. The following definitions and statements are due to S. Newhouse.

Fix $0 \le \gamma < 1$ and let $I = [-1,1]$, for $\psi: I \to I$ a C^1 map with $|D\psi| \le \gamma$ we say that $\{(\psi(y),y)\}$ ($\{(x,\psi(x))\}$) is a <u>u-curve</u> (s-curve). Given $\psi_1 \le \psi_2$ u-curves (s-curves) we shall call the set $V = \{(x,y) \in I^2 : \psi_1(y) \le x \le \psi_2(y)\}$ ($H = \{(x,y) \in I^2 : \psi_1(x) \le y \le \psi_2(x)\}$) a u-<u>rectangle</u> (s-rectangle). We shall say that $R_x \subset M$ is a <u>rectangle</u> in M if there exists a C^1 embedding Ψ such that $\Psi(I^2) = R_x$ and $\Psi(0,0) = x$, if U is a u-rectangle in I^2 we shall call $\Psi(U)$ a u-<u>rectangle in</u> R_x.

<u>Definition.</u>

<u>A</u> (ρ,λ)-<u>rectangle cover</u> of a set $\Lambda \subset M$ for $\rho > 0$, $\lambda > 1$ is a finite collection of rectangles on M $\{R_{x_1}, R_{x_2}, \ldots, R_{x_t}\}$ satisfying:

i) $\Lambda \subset \bigcup_{i=1}^{t} B,(x_i,\rho)$, $B(x_i,\rho) \subset \operatorname{int} R_{x_i}$ and $x_i \in \Lambda$.

ii) If $x \in \Lambda$, $f^n(x) \in \Lambda$ for some $n > 0$, $x \in B(x_i,\rho)$ and $f^n(x) \in B(x_j,\rho)$, then the connected component of $R_{x_i} \cap f^{-n}R_{x_j}$ containing x, that we denote by $C(x, R_{x_i} \cap f^{-n}R_{x_j})$, is an s-rectangle in R_{x_i} and $f^n((C(x, Rx_i \cap f^{-n}R_{x_j}))$ is a u-rectangle in R_{x_j}.

iii) $\operatorname{diam} f^m((C(x,R_{x_i} \cap f^{-n}R_{x_j})) \le 3 \operatorname{diam} R_{x_i} \max\{\lambda^{-m}, \lambda^{-(n-m)}\}$ for $0 \le m \le n$.

Theorem 2.1.8.

Let $f:M \to M$ be a C^2 diffeomorphism of a surface M preserving an ergodic Borel probability measure μ with non-zero exponents, then for any $\rho > 0$ there exists a compact set Λ with measure near to 1 which admits a (ρ,λ)-rectangle cover of arbitrary small diameters and $\lambda = \lambda(\Lambda)$.

Note.

The formulation of the above theorem is due to S. Newhouse, he attributes it to Katok and as we shall see all the elements for a proof are contained in [13] and have been quoted in the preceding pages.

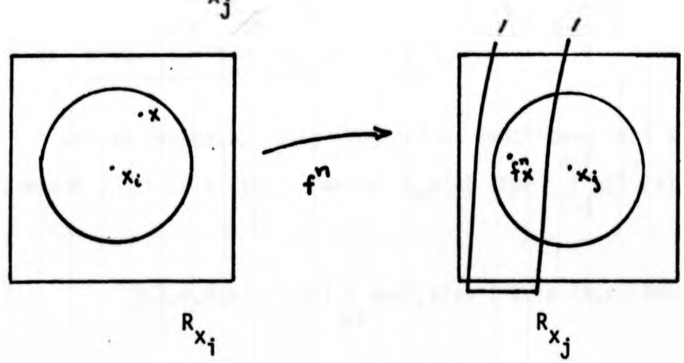
Sketch of the proof.

If the Lyapunov exponents are of the same sign then μ is supported in a hyperbolic periodic orbit [13], [31] and the proof follows from its hyperbolicity.

If $\chi_1 < 0 < \chi_2$ then $k = 1$ and we shall omit the dependence of any constant on it. So let $\chi = \min\{|\chi_1|,|\chi_2|\}$. By Corollary 2.1.3 choose $\ell > 1$ such that $\mu(\Lambda_{\chi,\ell})$ is as near to 1 as desired and let $\rho = \psi(\chi,h,\ell)$ of Proposition 2.1.7 for some $0 < h < 1$. Since $\Lambda_{\chi,\ell}$ is compact we can take a finite of it by balls of radius ρ , say $\{B(x_i,\rho)\}_{i=1}^{t}$ with $x_i \in \Lambda_{\chi,\ell}$. Set $R_{x_i} = \Phi_{x_i}[B^1_{h\varepsilon/4} \times B^1_{h\varepsilon/4}]$, then $B(x_i,\rho) \subset \operatorname{int} R_{x_i}$ and $\{R_{x_1},\dots,R_{x_t}\}$ satisfies i) of the definition of (ρ,λ)-rectangle cover.

Condition iii) follows from Propositions 2.1.4 and 2.1.5 with $\lambda = (\frac{1}{2} + \frac{1}{2\lambda(x)})$. Finally to check ii) we use Proposition 2.1.7 as follows:

Suppose that $x \in B(x_i,\rho) \cap \Lambda_{x,\ell}$ and for some $n > 0$ $f^n(x) \in B(x_j,\rho) \cap \Lambda_{x,\ell}$. Let $B_0 = \Phi_{x_i}\{(h\varepsilon/4,y) \mid y \in B^1_{h\varepsilon/4}\}$, since $x \in B(x_i,\rho)$ it follows that B_0 is an admissible (u,h)-manifold near x , then by Proposition 2.1.7 B_1 (as in the proposition) is an admissible (u,h)-manifold near x_j . Applying the same argument to $\Phi_{x_i}\{(-h\varepsilon/4,y) \mid y \in B^1_{h\varepsilon/4}\}$ we obtain that $C(f^n x, f^n R_{x_i} \cap R_{x_j})$ is a u-rectangle in R_{x_j} .



Again the same argument shows that $f^{-n}C(f^n x, f^n R_{x_i} \cap R_{x_j}) =$ $= C(x, R_{x_i} \cap f^{-n}R_{x_j})$ is an s-rectangle in R_{x_i} . □

§2. Measure-theoretic pressure.

Let $T:X \to X$ be a homeomorphism of a compact metric space X and μ a T-invariant Borel measure on X.

If d denotes the metric of X, let

$$d_n(x,y) = \max_{0\le i<n} d(T^i(x),T^i(y)) ,$$

for any $x,y \in X$. $d_n(\cdot,\cdot)$ is a metric on X and we shall call it the d_n-metric. Denote by $B_n(x,\varepsilon)$ the ε-ball centred at x in the d_n-metric.

For $\varepsilon > 0$ a set $E \subset X$ is said to be (n,ε)-spanning if $\bigcup_{x\in E} B_n(x,\varepsilon) \supset X$. Similarly for $\delta > 0$, $\varepsilon > 0$ a set $E \subset X$ is said to be μ-(n,ε,δ)-spanning if $(\bigcup_{X\in E} B_n(x,\varepsilon)) \ge 1-\delta$.

Let us denote by $C(X)$ the set of continuous real valued functions $\phi:M \to \mathbb{R}$. If $\phi \in C(X)$, write $S_n\phi(x)$ for $\sum_{i=0}^{n-1} \phi(T^i(x))$. Define

$$Q(T,\phi,n,\varepsilon) = \inf\{ \sum_{x\in E} \exp S_n\phi(x) \mid E \text{ is } (n,\varepsilon)\text{-spanning}\}$$

and for $\delta > 0$

$$Q_\mu(T,\phi,n,\varepsilon,\delta) = \inf\{ \sum_{x\in E} \exp S_n\phi(x) \mid E \text{ is } \mu\text{-}(n,\varepsilon,\delta)\text{-spanning}\} .$$

The topological pressure of T is defined as the map $P(T,\cdot):C(X) \to \mathbb{R}$, where

$$P(T,\phi) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q(T,\phi,n,\varepsilon) \ .$$

Similarly the measure-theoretic pressure of T with respect to μ is defined by

$$P_\mu(T,\phi) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q_\mu(T,\phi,n,\varepsilon,\delta) \ .$$

It is easy to check that for $m \geq 0$

$$P(T^m, S_m\phi) = mP(T,\phi) \ ,$$

$$P_\mu(T^m, S_m\phi) = mP_\mu(T,\phi) \ ,$$

$$P(T,0) = h(T)$$

and if μ is ergodic by Theorem 1.1.1

$$P_\mu(T,0) = h_\mu(T) \ .$$

In [38] Walters proved what is called the Variational Principle:

$$P(T,\phi) = \sup_{\substack{\mu \\ T\text{-invariant}}} \{h_\mu(T) + \int \phi \, d\mu\} \ .$$

We shall use this result to prove the following statement.

Theorem 2.2.1.

Let $T:X \to X$ be a homeomorphism of a compact metric space X, then for $\phi \in C(X)$

$$P(T,\phi) = \sup_{\substack{\mu \\ T\text{-invariant}}} P_\mu(T,\phi) .$$

Note.

The proof uses the same arguments of the proof of Theorem 1.1.1, that is those of Misiurewicz's proof of the Variational Principle [22], see also [37].

Proof.

It suffices to prove that $h_\mu(T) + \int \phi d\mu \le P_\mu(T,\phi)$, then by Walters' Variational Principle the theorem will follow.

As in Theorem 1.1.1, if ξ is a finite measurable partition of X, say $\xi = \{A_1,\ldots,A_k\}$, choose $B_i \subset A_i$ compact such that $\eta = \{B_0,B_1,\ldots,B_k\}$, where $B_0 = X \setminus \bigcup_{i=1}^{k} B_i$, has $H_\mu(\xi|\eta) < 1$.

For $\alpha > 0$, set

$$Y_N = \{y \in X \mid -\frac{1}{n}\log\mu(\eta_n(y)) \ge h_\mu(T,\eta)-\alpha \quad \forall n \ge N \quad \text{and}$$
$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(T^i y) \ge \int \phi d\mu - \alpha \qquad \forall n \ge N\} .$$

A combination of the S.M.B. theorem, Birkhoff's Ergodic Theorem and Egorov's theorem implies that for large N, $\mu(Y_N) > 0$.

Choose $\varepsilon > 0$ such that i) $2\varepsilon < b = \min d(B_i, B_j)$ $i \neq j \geq 1$, ii) $d(x,y) < \varepsilon \Rightarrow |\phi(x)-\phi(y)| < \alpha$. Then $\mu(B_n(x,\varepsilon) \cap Y_N) \leq$ $\leq \exp n(\log 2 - h_\mu(T,\eta)+\alpha)$, since $B_n(x,\varepsilon) \cap Y_N$ can be covered by at most 2^n elements of η

Let E be a μ-(n,ε,δ)-spanning set for $n \geq N$ and $0 < \delta \ll \mu(Y_N)$, consider $E' = \{x \in E \mid B_n(x,\varepsilon) \cap Y_N \neq \phi\}$. By continuity if $y(x) \in B_n(x,\varepsilon) \cap Y_N$ then $S_n(\phi(x)) - S_n(\phi(y(x))) \geq -\alpha n$. Therefore it follows that

$$\sum_{x\in E} \exp S_n\phi(x) \exp -n\left(\int \phi d\mu - 3\alpha - \log 2 + h_\mu(T,\eta)\right) \geq$$

$$\sum_{x\in E'} \exp\left(S_n\phi(x) - n\int \phi d\mu\right) \exp -n(-3\alpha - \log 2 + h_\mu(T,\eta)) =$$

$$\sum_{x\in E'} \exp\left(S_n\phi(x) - S_n\phi(y(x)) + S_n\phi(y(x)) - n\int \phi d\mu\right) \exp -n(-3\alpha - \log 2 + h_\mu(T,\eta)) \geq$$

$$\sum_{x\in E'} \exp -\alpha n \exp -\alpha n \exp 2\alpha n \exp -n(-\alpha - \log 2 + h_\mu(T,\eta)) =$$

$$\sum_{x\in E'} \exp n\,(\log 2 + \alpha - h_\mu(T,\eta)) \geq$$

$$\sum_{x\in E'} \mu(B_n(x,\varepsilon) \cap Y_N) \geq \mu\left(\bigcup_{x\in E'} B_n(x,\varepsilon)\right) > 0 ,$$

which implies that

$$Q_\mu(T,\phi,\varepsilon,\delta) \geq h_\mu(T,\eta) + \int \phi d\mu - \log 2 - 3\alpha \quad .$$

Since α and ξ are arbitrary, and $h_\mu(T,\xi) \leq h_\mu(T,\eta) + H_\mu(\xi|\eta)$

$$P_\mu(T,\phi) \geq h_\mu(T) + \int \phi d\mu - \log 2 - 1 \; .$$

Now apply the above procedure to T^m and $S_m\phi$, to obtain

$$P_\mu(T,\phi) \geq \frac{1}{m}\left(h_\mu(T^m) + \int S_m\phi d\mu - \log 2 - 1\right)$$

so letting $m \to \infty$

$$P_\mu(T,\phi) \geq h_\mu(T) + \int \phi d\mu \quad . \qquad \square$$

<u>Remark</u>.

The sup in Theorem 2.2.1 can be taken over the ergodic measures.

§3. <u>On the prevalence of horseshoes</u>.

We borrow the title of this section from L.S. Young, because basically we are going to generalize Theorem 1 of [40] for the pressure of two dimensional C^2 diffeomorphisms.

We recall the definition of a <u>horseshoe</u> [23], [36]. Let $I = [-1,1]$, fix $0 \le \gamma < 1$ and consider a finite collection of disjoint u-rectangles $\{V_i\}_{i=1}^{t}$ in I^2 such that the u-curves that defined each rectangle have derivative less than or equal to γ. Similarly, consider a finite collection of disjoint s-rectangles $\{H_i\}_{i=1}^{t}$ in I^2.

Let $T: I^2 \to \mathbb{R}^2$ be a continuous map satisfying the following conditions:

i) For each $0 \le i \le t$, $T|H_i : H_i \to V_i$ is a homeomorphism.

ii) The intersection $H_i \cap \{(1,y) \mid y \in [-1,1]\}$ is mapped by T onto one of the u-curves defining V_i, and $H_i \cap \{(-1,y) \mid y \in [-1,1]\}$ is mapped onto the other u-curve defining V_i. The s-curves defining H_i are mapped into $\{(x,1) \mid x \in [-1,1]\} \cup \{(x,-1) \mid x \in [-1,1]\}$.

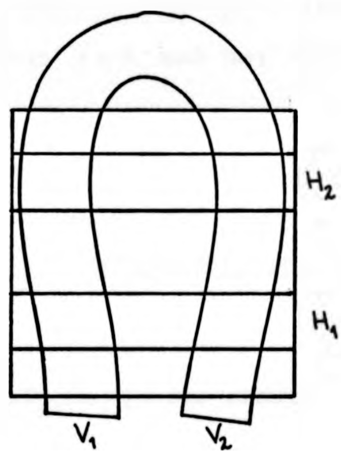
iii) There exists $0 < \nu < 1$ such that if H is an s-rectangle contained in $\bigcup_{i=1}^{t} H_i$ then for any $0 \le i \le t$,

$$T^{-1}(H) \cap H_i = \tilde{H}_i$$

is a non-empty s-rectangle and $d(\tilde{H}_i) \le \nu d(H_i)$, with $d(H) = \max_{-1 \le x \le 1} |v_2(x) - v_1(x)|$ where v_1, v_2 are the defining s-curves of H, an s-rectangle.

iv) Similar conditions to ii) and iii) are satisfied for u-rectangles.

A quadruple $(T, t, \{H_i\}_{i=1}^{t}, \{V_i\}_{i=1}^{t})$ satisfying the above conditions for γ and ν is called a <u>horseshoe with parameters</u> (γ, ν) .



<u>Theorem 2.3.1</u>. [23]

<u>Let</u> $T: I^2 \to \mathbb{R}^2$ <u>be a continuous map satisfying the above conditions for a horseshoe and</u> $\Lambda = \bigcap_{n=-\infty}^{\infty} T^n(\bigcup_{i=1}^{t} H_i)$. <u>Then there exists a homeomorphism</u> $\phi: \Lambda \to \Sigma_t$ <u>such that</u> $\phi T = \sigma\phi$, <u>where</u> $\sigma: \Sigma_t \to \Sigma_t$ <u>is the two-sided shift in</u> t <u>symbols</u>. □

Now suppose that $f:M \to M$ is a diffeomorphism of a surface M and let R be a rectangle on M defined by $\Phi:I^2 \to R$. If for some $n > 0$ f^n maps t disjoint s-rectangles in R onto t disjoint u-rectangles in R and $\Phi^{-1} \circ f^n \circ \Phi$ satisfies conditions i) to iv) of the definition of a horseshoe for some parameters (γ,ν) , then we say that f has a t-folds horseshoe of period n inbedded in R . By Theorem 2.3.1 there exists $\Lambda \subset R$ such that $f^n|\Lambda$ is conjugate to $\sigma:\Sigma_t \to \Sigma_t$.

Theorem 2.3.2.

Let $f:M \to M$ *be a* C^2 *diffeomorphism of a surface* M *and* $\phi \in C(M)$ *such that*

$$P(f,\phi) = \sup\{P_\mu(f,\phi) \mid \mu \text{ has nonzero exponents}\} ,$$

then

$$P(f,\phi) = \sup\{P(f|\Omega,\phi|\Omega) \mid \Omega \text{ is a hyperbolic set}\} ,$$

moreover the sup can be taken over zero topological dimensional sets.

Proof.

For $\alpha > 0$ choose an ergodic f-invariant measure μ with non-zero exponents such that $P_\mu(f,\phi) \geq P(f,\phi)-\alpha$. If the exponents $\chi_1^\mu \leq \chi_2^\mu$ are of the same sign then μ is supported in a hyperbolic periodic orbit (a sink or a source depending on the sign of the exponents) $\theta(p)$ for $p \in M$ and $P_\mu(f,\phi) = P(f|\theta(p),\phi|\theta(p))$.

If $\chi_1^\mu < 0 < \chi_2^\mu$, let $\delta > 0$, $\epsilon > 0$ be such that

$$\limsup_{n\to\infty} \frac{1}{n} \log Q\,(n,\epsilon,\delta) \geq P_\mu(f,\phi)-\alpha$$

and if $d(x,y) < \epsilon$ then $|\phi(x)-\phi(y)| < \alpha$.

By Theorem 2.1.8 we can choose a set $\Lambda \subset M$ such that $\mu(\Lambda) > 1-\delta/2$, for $\rho > 0$ small and $\lambda = \lambda(\Lambda) > 1$ the set Λ admits a (ρ,λ)-rectangle cover $\{R_{x_1},\ldots,R_{x_t}\}_{j=1}^{t}$ such that diam $R_j < \epsilon/3$. Now, let ξ be a finite measurable partition of M with diam $\xi < \rho/2$ and

$$\Lambda_n = \{x \in \Lambda : f^q(x) \in \xi(x) \text{ for some } q \in [n,(1+\alpha)n]\} \quad .$$

Lemma 2.3.3. [13]

$\mu(\Lambda_n) \to \mu(\Lambda)$ as $n \to \infty$.

Proof.

Let $B \in \xi$ and set

$$B_{n,\alpha} = \{x \in B : \frac{1}{n}\sum_{k=0}^{n-1} \chi_B(f^k(x)) < \mu(B)(1+\alpha/3) \text{ and}$$

$$\mu(B)(1+2\alpha/3) < \frac{1}{n}\sum_{k=0}^{[n(1+\alpha)]} \chi_B(f^k(x))\} \quad ,$$

periodic orbit (a sink or a source depending on the sign of the exponents) $\theta(p)$ for $p \in M$ and $P_\mu(f,\phi) = P(f|\theta(p),\phi|\theta(p))$.

If $\chi_1^\mu < 0 < \chi_2^\mu$, let $\delta > 0$, $\varepsilon > 0$ be such that

$$\limsup_{n\to\infty} \frac{1}{n} \log Q\,(n,\varepsilon,\delta) \geq P_\mu(f,\phi)-\alpha$$

and if $d(x,y) < \varepsilon$ then $|\phi(x)-\phi(y)| < \alpha$.

By Theorem 2.1.8 we can choose a set $\Lambda \subset M$ such that $\mu(\Lambda) > 1-\delta/2$, for $\rho > 0$ small and $\lambda = \lambda(\Lambda) > 1$ the set Λ admits a (ρ,λ)-rectangle cover $\{R_{x_1},\ldots,R_{x_t}\}_{j=1}^t$ such that $\operatorname{diam} R_j < \varepsilon/3$. Now, let ξ be a finite measurable partition of M with $\operatorname{diam}\ \xi < \rho/2$ and

$$\Lambda_n = \{x \in \Lambda : f^q(x) \in \xi(x) \text{ for some } q \in [n,(1+\alpha)n]\} \ .$$

<u>Lemma 2.3.3</u>. [13]

$\mu(\Lambda_n) \to \mu(\Lambda)$ <u>as</u> $n \to \infty$.

<u>Proof</u>.

Let $B \in \xi$ and set

$$B_{n,\alpha} = \{x \in B : \frac{1}{n}\sum_{k=0}^{n-1} \chi_B(f^k(x)) < \mu(B)(1+\alpha/3) \text{ and}$$

$$\mu(B)(1+2\alpha/3) < \frac{1}{n}\sum_{k=0}^{[n(1+\alpha)]} \chi_B(f^k(x))\} \ ,$$

where χ_B is the characteristic function on the set B .

By the Ergodic Theorem $\mu(B \setminus B_{n,\alpha}) \to 0$ as $n \to \infty$, which implies that $\mu(\Lambda_n) \to \mu(\Lambda)$. □

So for large n , $\mu(\Lambda_n) > 1-\delta$. Let $E_n \subset \Lambda_n$ be an (n,ε)-<u>separated</u> set of maximal cardinality (i.e. $\forall\, x \neq y \in E_n$ there exists $k \in [0,n)$ such that $d(f^k(x),f^k(y)) > \varepsilon$ and E_n is a maximal subset of Λ_n having this property), clearly $\bigcup_{x \in E_n} B_n(x,\varepsilon) \supset \Lambda_n$ and therefore there exist infinitely many n's such that

$$\sum_{x \in E_n} \exp S_n\phi(x) > \exp n(P_\mu(f,\phi)-2\alpha) .$$

For each $q \in [n,(1+\alpha)n]$ let $F_q = \{x \in E_n \mid f^q(x) \in \xi(x)\}$, and now let m be the value of q that maximises $\sum_{x \in F_q} \exp S_n\phi(x)$, since $\exp n\alpha \geq n\alpha$

$$\sum_{x \in F_m} \exp S_n\phi(x) \geq \exp n(P_\mu(f,\phi)-3\alpha) .$$

Consider $F_m \cap R_{x_j}$ for $1 \leq j \leq t$ and choose the value i of j that maximizes $\sum_{x \in F_m \cap R_{x_j}} \exp S_n\phi(x)$. Thus if we write D_m for $F_m \cap R_{x_i}$

$$\sum_{x\in D_m} \exp S_n\phi(x) \geq \frac{1}{t} \sum_{x\in F_m} \exp S_n\phi(x) \geq \frac{1}{t} \exp n(P_\mu(f,\phi)-3\alpha)) \ .$$

So consider R_{x_i} and D_m. Each $x \in D_m$ returns to R_{x_i} in m iterations, so $C(f^m(x),R_{x_i} \cap f^m R_{x_i})$ is a u-rectangle in R_{x_i} and $f^{-m}C(f^m(x),R_{x_i} \cap f^m R_{x_i})$ an s-rectangle. This follows from the facts that $d(x_i,x) < \rho$ and $d(f^m(x),x_i) < \rho$, and ii) of the definition of a (ρ,λ)-rectangle cover.

If $y \in C(x,R_{x_i} \cap f^{-m}R_{x_i})$ then by iii) of the definition of a (ρ,λ)-rectangle cover

$$d(f^k(x),f^k(y)) \leq \text{diam } f^k C(x,R_{x_i} \cap f^{-m}R_{x_i}) \leq 3 \text{ diam } R_{x_i}$$

$$\leq \varepsilon \quad \text{for } k \in [0,m) \ ,$$

which implies: i) $|S_m\phi(y)-S_m\phi(x)| \leq \alpha m$ and ii) that if $y \neq x$ and $y \in C(x,R_{x_i} \cap f^{-m}R_{x_i})$ then $y \notin D_m$, otherwise it would contradict the separability of D_m.

Hence there exist $\# D_m$ disjoint s-rectangles mapped by f^m onto $\# D_m$ u-rectangles. Using Propositions 2.1.5 and 2.1.7 it can be shown that conditions ii) and iii) of the definition of a horseshoe are satisfied. So let

$$\tilde{\Lambda} = \bigcap_{k=-\infty}^{\infty} f^{mk}\Big(\bigcup_{x\in D_m} C(x,R_{x_i} \cap f^{-m}R_{x_i})\Big)$$

by Theorem 2.3.1 $f^n|\tilde{\Lambda}$ is conjugate to $\sigma:\Sigma_{\# D_m} \to \Sigma_{\# D_m}$.

Similar arguments to the ones used by Katok [13] (page 163) show that $\tilde{\Lambda}$ is a hyperbolic set.

Let $\Omega = \bigcup_{k=0}^{m} f^k\tilde{\Lambda}$. To estimate $P(f|\Omega,\phi|\Omega)$ we shall use the symbolic dynamic of $f^m|\tilde{\Lambda}$ and the fact [31] that

$$P(f|\Omega,\phi|\Omega) = \limsup_{k\to\infty} \frac{1}{k} \log \sum_{z\in \mathrm{Fix} f^k \cap \Omega} \exp S_k(z) ,$$

where $\mathrm{Fix}\, f^k = \{x \in M \mid f^k(x) = x\}$.

So if $y \in \tilde{\Lambda}$ is periodic point of period $N = km$, then there exists a unique k-tuple $\overline{x} = \overline{x}(y) = (x^1,x^2,\ldots,x^k) \in D_m^k$ such that

$$d(f^j(y),f^j(x^1)) < \varepsilon \quad \text{for } j \in [0,m) ,$$

$$d(f^j(y),f^j(x^2)) < \varepsilon \quad \text{for } j \in [m,2m)$$

$$\vdots$$

$$d(f^j(y),f^j(x^k)) < \varepsilon \quad \text{for } j \in [(k-1)m,km) .$$

Therefore $S_N(\phi(y)+\alpha) \geq S_m\phi(x^1) + S_m\phi(x^2)+\ldots+ S_m\phi(x^k)$, and

$$\sum_{y \in \mathrm{Fix} f^N \cap \tilde{\Lambda}} \exp S_N(\phi(y)+\alpha) \geq \sum_{\bar{x}(y) \in D_m^k} \prod_{\ell=1}^{k} \exp S_m\phi(x^\ell) =$$

$$= \Big(\sum_{x \in D_m} \exp S_m\phi(x)\Big)^k .$$

Since $S_m\phi(x) \geq S_n\phi(x) + (m-n-1) \inf \phi$ and $N = km$

$$\frac{1}{k} \log \sum_{y \in \mathrm{Fix} f^{mk} \cap \tilde{\Lambda}} \exp S_{km}(\phi(y)+\alpha) \geq \log \sum_{x \in D_m} \exp S_m\phi(x) \geq$$

$$\log \sum_{x \in D_m} \exp S_n\phi(x) + (m-n-1)\inf \phi .$$

Which implies that letting $k \to \infty$

$$P(f|\Omega,(\phi+\alpha)|\Omega) \geq \frac{1}{m} \log \sum_{x \in D_m} \exp S_n\phi(x) + (m-n-1)\inf \phi \geq$$

$$\frac{n}{m} (P_\mu(f,\phi)-3\alpha) + (m-n-1)\inf \phi .$$

Now using the inequalities

$$\frac{1}{(1+\alpha)n} \leq \frac{m-n-1}{m} \leq \alpha - \frac{1}{n} \ , \quad \frac{1}{1+\alpha} \leq \frac{n}{m} \ ,$$

letting $n \to \infty$ and $\alpha \to 0$, we obtained that

$$\sup_{\Omega \text{ hyperbolic}} P(f|\Omega,\phi|\Omega) \geq P_\mu(\phi) .$$

□

<u>Corollary 2.3.4</u>.

If $f:M \to M$ *is a* C^2 *diffeomorphism of a surface* M *with* $h(f) > 0$, *then*

$$h(f) = \sup\{h(f|\Omega) \mid \Omega \text{ is a horseshoe}\} \quad .$$

□

CHAPTER 3.

Hausdorff Dimension of Generic Points.

§0. Introduction.

Let $T:X \to X$ be a homeomorphism of a compact metric space X preserving an ergodic Borel probability measure μ. We say that a point $x \in X$ is future generic or simply generic for μ if for any $\phi \in C(X)$

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(T^i(x)) \to \int \phi d\mu \quad ,$$

as $n \to \infty$. Let G_μ denote the set of generic points of μ, it is easy to prove that $\mu(G_\mu) = 1$.

In [33] Ruelle proved that if Λ is a hyperbolic attractor for a C^2 diffeomorphism f of a manifold M, then there exists a unique measure μ such that the set of generic points G_μ has positive Lebesgue measure.

Although a good deal of the work done in dynamical systems, nowadays, is directed towards the generalization of Ruelle's theorem, it remains an open problem. The aim of this chapter is to generalize part of Manning's theorem [18] to prove a very weak version of Ruelle's theorem. Namely, if $f:M \to M$ is a C^2 diffeomorphism of a surface M preserving an ergodic f-invariant Borel probability measure μ, then if $h_\mu(f) > 0$

$$HD(G_\mu) \geq 1 + h_\mu(f)/\chi_\mu^+ ,$$

where χ_μ^+ is the positive Lyapunov exponent of μ.

So if $h_\mu(f) = \chi_\mu^+$, then $HD(G_\mu) = 2$. By [17] this is the case for an important set of measures called absolutely continuous on the unstable leaves of f.

Finally, we study the measures supported on uniform hyperbolic sets by combining ideas similar to the ones used by Bowen [4] and McCluskey and Manning [19] with Theorem 2.3.2 of this thesis.

§1. Pesin's stable manifold theorem.

We shall follow the notation introduced in §1 of Chapter 2. Let $f:M \to M$ be a C^2 diffeomorphism of a compact N-manifold M. We recall that $\Lambda = \bigcup_{k,\ell,\chi} \Lambda^k_{\chi,\ell}$ and for each $x \in \Lambda$, $T_xM = E^s_x \oplus E^u_x$ with $k = \dim E^s_x$.

Theorem 3.1.1. (Stable Manifold Theorem) [29]

There exist a measurable function $\delta:\Lambda \to (0,1)$, a family of C^1 maps $\phi(x):B^k_{\delta(x)} \to B^{N-k}_{\delta(x)}$ depending measurably on $x \in \Lambda$ and a constant $C > 0$ satisfying the following conditions:

i) The set $W^s_{loc}(x) = \{\Phi_x(v,\phi(x)v) : v \in B^k_{\delta(x)}\}$ is a C^1 submanifold of M.

ii) For every $x \in \Lambda$, $x \in W^s_{loc}(x)$ and $T_x W^s_{loc}(x) = E^s_x$.

iii) For $y \in W^s_{loc}(x)$ and $n \geq 0$, if $x \in \Lambda_\chi$, then

$$d(f^n(x),f^n(y)) \leq CA(x)\exp(-\frac{99}{100}\chi n)d(x,y) .$$

iv) For any $m \geq 0$ and $x \in \Lambda_\chi$

$$\delta(x) \leq \min\{(3/2)^m, \exp 4\times10^{-3}m)\delta(f^m(x))\} ,$$

$\delta(x) \leq \varepsilon(x)$ and

$$\delta^k_{\chi,\ell} = \inf_{x\in\Lambda^k_{\chi,\ell}} \delta(x) > 0 .$$

v) $f(W^s_{loc}(x)) \cap \Phi_{f(x)}[B^k_{\delta(f(x))} \times B^{N-k}_{\delta(f(x))}] \subseteq W^s_{loc}(f(x))$. □

We recall that Φ_x is the Lyapunov chart, $\varepsilon(x)$ is the size of the reduced neighbourhood $C(x)$, and $A(x)$ measures the distortion between the Lyapunov metric and the given Riemannian metric of the manifold. See Proposition 2.1.4.

The submanifold $W^s_{loc}(x)$ is called the local stable manifold through x. If we apply the above theorem to f^{-1} we get $W^u_{loc}(x)$,

<u>the local unstable manifold through</u> x . Let us denote by $W^s(x)$ the set

$$\{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0\} \quad .$$

In [32] Ruelle proved that $W^s(x) = \bigcup_{i=0}^{\infty} f^{-i} W^s_{loc}(f^i(x))$, hence $W^s(x)$ is an immersed C^1 submanifold. We shall call $W^s(x)$ <u>the global stable manifold through</u> x . Similarly define $W^u(x)$, <u>the global unstable manifold through</u> x .

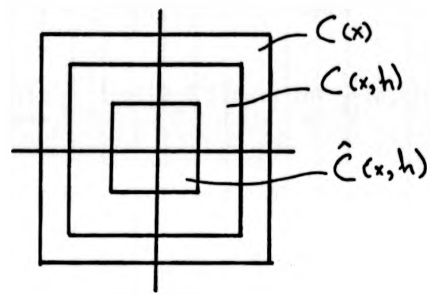
Now we concentrate our attention to the case $\dim M = 2$ and $k = \dim E^s_x = 1$. For convenience we shall ignore the dependence on k of some constants and notation, for instance $\Lambda_{\chi,\ell}$ will denote $\Lambda^1_{\chi,\ell}$.

<u>Remark 3.1.2</u>.

We may assume that $\delta(x) = \varepsilon(x)$ and that $0 < h \leq 1/4$ is so small that for any $x, y \in \Lambda_{\chi,\ell}$ with $y \in \hat{C}(x,h)$, then $W^s_{loc}(y)$ is an admissible $(s,1)$-manifold near x . This is possible by Corollary 2.1.6. So fix $h = h(\chi,\ell) \in (0,1/4]$, the collection of local stable manifolds passing through $\hat{C}(x,h) \cap \Lambda_{\chi,\ell}$ is called <u>the family of local stable manifolds</u> $S_{\chi,\ell}(x)$. By Proposition 2.5 of [13] if V is an admissible $(u,1)$-manifold near x , then V is transverse to $S_{\chi,\ell}(x)$.

□

Let us recall that $C(x) = \Phi_x[B^1_{\varepsilon(x)} \times B^1_{\varepsilon(x)}]$, $C(x,h) = \Phi_x[B^1_{h\varepsilon(x)} \times B^1_{h\varepsilon(x)}]$ and for $\varepsilon = \varepsilon(X,\ell) \leq \varepsilon(x)$, $\hat{C}(x,h) = \Phi_x[B^1_{h\varepsilon/2} \times B^1_{h\varepsilon/2}]$.



§2. Generic points of ergodic measures of diffeomorphisms of surfaces.

In this section we prove the main result of this chapter.

Theorem 3.2.1.

Let $f: M \to M$ be a C^2 diffeomorphism of a surface M preserving an ergodic Borel probability measure μ . If μ has nonzero Lyapunov exponents of different signs, then

$$HD(G_\mu) \geq 1 + h_\mu(f)/\chi^+_\mu ,$$

where χ^+_μ is the positive Lyapunov exponent of μ .

Proof.

We have adopted the notation used in Chapter 1 for the positive Lyapunov exponent of μ, because it will appear at the end of the proof as an application of the Subadditive Ergodic Theorem [14] to the sequence of functions $x \to \frac{1}{n} \log||D_x f^n||$. Hence

$$\chi_\mu^+ = \chi_2^\mu = \lim_{n\to\infty} \frac{1}{n} \int \log||D_x f^n|| d\mu = \inf_n \frac{1}{n} \int \log||D_x f^n|| d\mu ,$$

where $\chi_1^\mu < 0 < \chi_2^\mu$ are the Lyapunov exponents of μ as defined in Chapter 2.

Let $\chi = \min\{-\chi_1^\mu , \chi_2^\mu\}$ and choose $\ell > 1$ such that $\mu(\Lambda_{\chi,\ell}) > 0$. Denote by $\Lambda_{\chi,\ell,\mu}$ the set of all $x \in \Lambda_{\chi,\ell}$ such that for any $\rho > 0$, $\mu(B(x,\rho) \cap \Lambda_{\chi,\ell}) > 0$, we shall call such a point x a μ-density point for $\Lambda_{\chi,\ell}$. Clearly $\mu(\Lambda_{\chi,\ell,\mu}) = \mu(\Lambda_{\chi,\ell})$. Now take $x \in \Lambda_{\chi,\ell,\mu}$ such that $\mu(C\ell\{f^n(x) \in \Lambda_{\chi,\ell} \mid n \geq 0\}) = \mu(\Lambda_{\chi,\ell})$, this is possible by the following statement.

Lemma 3.2.2.

Let $T:X \to X$ be a homeomorphism of a compact metric space X preserving an ergodic Borel probability measure μ. If $A \subset X$ is a closed set of positive μ-measure, then

$$\mu(\{x \in A \mid \mu(C\ell\{T^n(x) \in A \mid n \geq 0\}) = \mu(A)\}) = \mu(A) .$$

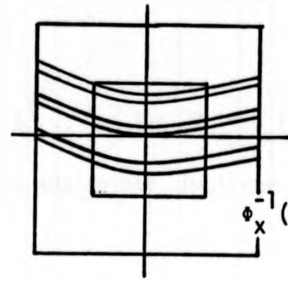
Proof.

By Poincaré's Recurrence Theorem there exists $B \subset A$ such that for any $x \in B$ the orbit of x returns infinitely many times to B and $\mu(B) = \mu(A)$. We may assume that any $x \in B$ is a μ-density point for A.

Let $\psi_T : B \to B$ be the first return map to B, i.e. $\psi_T(x) = T^{n(x)}(x)$ where $n(x)$ is the smallest positive integer satisfying $T^{n(x)}(x) \in B$. Then the conditional measure on B, μ_B, is ergodic and ψ_T-invariant.

So let us consider B as a metric space with the induced metric and let $\{U_n\}_{n=1}^{\infty}$ be a countable base for the topology of B. By the definition of B each U_n has positive μ_B-measure and by ergodicity $\bigcup_{k=0}^{\infty} \psi_T^{-k} U_n$ has full measure on B. Therefore the set $Y = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \psi_T^{-k} U_n$ has μ_B-measure 1 and if $x \in Y$ then the ψ_T-orbit is dense in B. But since A is closed, $C\ell\{T^n(x) \in A \mid n \geq 0\} \subset A$, from where the lemma follows. $\square$

Back to the main argument, the point $x \in \Lambda_{X,\ell,\mu}$ such that $\mu(C\ell\{f^n(x) \in \Lambda_{X,\ell} \mid n \geq 0\}) = \mu(\Lambda_{X,\ell})$ will remain fixed for the rest of the proof. Now let $h = h(X,\ell)$ be as in Remark 3.1.2 and consider the family of local stable manifolds $S_{X,\ell}(x)$. Observe that if $z \in W^s(y)$ and $y \in \Lambda_{X,\ell} \cap G_\mu$, then $z \in G_\mu$. We shall write $S_{X,\ell,\mu}(x)$ for $S_{X,\ell}(x) \cap G_\mu$.



Having the picture in mind we shall describe briefly the idea of the proof. To estimate from below the Hausdorff dimension of a set like $\Phi_x^{-1}(S_{x,\ell,\mu}(x))$ we can use a sort of Fubini's Theorem for Hausdorff dimension of plane sets due to Marstrand. To apply this theorem we need to know about the dimension of $\Phi_x^{-1}(S_{x,\ell,\mu}(x))$ intersected with any vertical line. These two steps are formalized in the following statements, from which the Theorem follows.

Proposition 3.2.3. [21]

Suppose that E is a plane set, write $E_x = \{(x,y) \in E \mid y \in \mathbb{R}\}$, also suppose that $c > 0$ is such that for every point x of a given set $A \subset \{(x,0) | x \in \mathbb{R}\}$ we have that $m_t(E_x) > c$. Then

$$m_{s+t}(E) \geq K\, c\, m_s(A)$$

where K is a positive absolute constant. □

Here $m_\lambda(\cdot)$ denotes the λ-Hausdorff measure as defined in §2 of Chapter 1.

For $r \in B^1_{h\varepsilon/4}$, let $V_r = \Phi_x\{(r,v) | v \in B^1_\varepsilon\}$. It is obvious that V_r is an admissible $(u,1)$-manifold near x , and hence transverse to $S_{x,\ell}(x)$.

Proposition 3.2.4.

For any $r \in B^1_{h\varepsilon/4}$, $h_\mu(f) \le HD(G_\mu \cap V_r)X^+_\mu$.

Before proving this proposition, let us show how we can use it to prove Theorem 3.2.1.

Let $\alpha > 0$ be arbitrary and $r \in B^1_{h\varepsilon/4}$. If $\lambda = h_\mu(f)/X^+_\mu - \alpha$, then by Proposition 3.2.4

$$m_\lambda(\Phi_x^{-1}(V_r \cap G_\mu)) > 1 ,$$

hence by Proposition 3.2.3.

$$m_{1+\lambda}(\Phi_x^{-1}(G_\mu \cap \hat{C}(x,h/2))) \ge Kh\varepsilon/2 > 0 .$$

Therefore $HD(\Phi_x^{-1}(G_\mu \cap \hat{C}(x,h/2))) \ge 1 + h_\mu(f)/X^+_\mu - \alpha$. Since α is arbitrary and Φ_x is a diffeomorphism we have that

$$HD(G_\mu) \ge HD(G_\mu \cap \hat{C}(x,h/2)) \ge 1 + h_\mu(f)/X^+_\mu . \qquad \square$$

Now we proceed to prove Proposition 3.2.4. The idea behind the proof is similar to one used by Manning in [18]. Namely to cover $\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2)$ by thickened admissible $(u,1)$-manifolds near x. Using Bowen's definition of entropy for non-compact sets, see Chapter 1, and the fact that $||D_z f||$ is continuous on M we show that

$$h_\mu(f) \le HD(G_\mu \cap V_r) \int \log||D_z f||\,d\mu$$

for any $r \in B^1_{h\varepsilon/4}$. Then by the Subadditive Ergodic Theorem the proof of Theorem 3.2.1 follows.

If V is a transverse submanifold to $S_{\chi,\ell}(x)$ write $W^s(V,t)$ for $\bigcup_{z \in V \cap S_{\chi,\ell}(x)} D(z,t)$, where $D(z,t)$ denotes the 1-disk of radius t centred at z on $W^s_{loc}(y)$, $y \in \Lambda_{\chi,\ell} \cap \hat{C}(x,h/2)$ and $z = z(y)$ is the point of intersection of $W^s_{loc}(y)$ with V. So let us fix $r \in B^1_{h\varepsilon/4}$ and consider V_r.

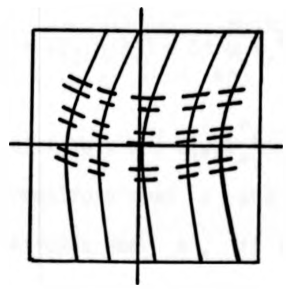
We now select some iterates of V_r that when thickened will give us a cover of $\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2)$.

Lemma 3.2.5.

There exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of nonnegative integers such that given any $t > 0$ there exists $N \ge 0$ such that

$$\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2) \subset \bigcup_{k=0}^{N} W^s(C(f^{n_k}(z(x)), f^{n_k}V_r \cap \hat{C}(x,1)), t) ,$$

<u>where</u> $z(x)$ <u>is the point of intersection of</u> V_r <u>with</u> $W^s_{loc}(x)$.



<u>Proof</u>.

Let $w \in \Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2)$, by the choice of x there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of nonnegative integers such that $f^{n_j}(x) \in \Lambda_{\chi,\ell}$ and $\lim_{j\to\infty} f^{n_j}(x) = w$ and since $z(x) \in W^s_{loc}(x)$, then $\lim_{j\to\infty} f^{n_j}(z(x)) = w$. So for large j's , $d(f^{n_j}(x),w) < \psi(\chi,\ell)$.

By Proposition 2.17

$$V_r^{n_j} = C(f^{n_j}(z(x)),f^{n_j}V_r \cap \hat{C}(x,1))$$

is an admissible $(u,1)$-manifold near $f^{n_j}(x)$. Hence $V_r^{n_j}$ is an admissible $(u,1)$-manifold near w , but since $w \in \hat{C}(x,h/2)$ then $V_r^{n_j}$ is an admissible $(u,1)$-manifold near x . Furthermore, if $z_{n_j}(w)$ is the point of intersection of $V_r^{n_j}$ with $W^s_{loc}(w)$, then $z_{n_j}(w) \to w$ as $j \to \infty$.

Therefore there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of nonnegative integers such that

$$C\ell(\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2)) \subset C\ell \bigcup_{k=1}^{\infty} V_r^{n_k} .$$

Now let $B(V_r^{n_k},t) = \{w \in M \mid d(w,V_r^{n_k}) < t\}$, since $V_r^{n_k}$ is an admissible $(u,1)$-manifold near x and $S_{\chi,\ell}(x)$ is formed by admissible $(s,1)$-manifolds near x , it is straightforward to check (using the Lyapunov chart at x) that for $K_3 = \min\{\frac{1}{2},\sqrt{2}K_1K_2A_{\chi,\ell}(1-\gamma^2)\}$ and $t > 0$ we have

$$S_{\chi,\ell}(x) \cap B(V_r^{n_k},K_3t) \subset W^s(V_r^{n_k},t) \subset B(V_r^{n_k},t) ,$$

where K_1,K_2, $A_{\chi,\ell}$, γ are as in Proposition 2.1.4.

By compactness given $t > 0$ there exists $N > 0$ such that

$$\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2) \subset \bigcup_{k=1}^{N} B(V_r^{n_k},K_3t) ,$$

and clearly

$$\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2) \cap S_{\chi,\ell}(x) = \Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2) .$$

Hence

$$\begin{aligned}\Lambda_{\chi,\ell,\mu} \cap \hat{C}(x,h/2) &\subset \bigcup_{k=1}^{N} B(V_r^{n_k},K_3t) \cap S_{\chi,\ell}(x) \\ &\subset \bigcup_{k=1}^{N} W^s(V_r^{n_k},t) . \qquad \square\end{aligned}$$

The above lemma gives us a way of constructing covers of $\Lambda_{X,\ell,\mu} \cap \hat{C}(x,h/2) \cap G_\mu$ by thickening $V_r^{n_k}$'s . So if we cover each $V_r^{n_k} \cap S_{X,\ell,\mu}(x)$ by intervals contained in $V_r^{n_k}$ of arbitrary small diameters, then combining these covers with the thickened $V_r^{n_k}$'s we obtain new covers of $\Lambda_{X,\ell,\mu} \cap \hat{C}(x,h/2) \cap G_\mu$ of arbitrary small diameters. We shall use these covers to get an upper bound of $h(\Lambda_{X,\ell,\mu} \cap \hat{C}(x,h/2) \cap G_\mu, f)$ (which is $\geq h_\mu(f)$ by Corollary 1.1.4, since $x \in \Lambda_{X,\ell,\mu}$) in terms of $HD(V_r \cap G_\mu)$ and $\int \log||D_x f||d\mu$. For this we shall use Bowen's definition of entropy for noncompact sets discussed in Chapter 1.

For $\alpha > 0$, let A be a finite open cover of M such that $||D_x f||$ varies at most $\alpha > 0$ in each element of A and let L be a Lebesgue number for A . Fix $\beta > 0$ and for $n \geq 0$ set

$$G_{\mu,n} = \{y \in G_\mu : |\frac{1}{m}\sum_{k=0}^{m-1} \log(||D_{f^k(y)} f||+\alpha) - \int \log(||D_z f||+\alpha)d\mu| \leq \beta , \quad \forall\, m \geq n\} .$$

Clearly $G_\mu = \bigcup_n G_{\mu,n}$.

Fix $0 < t \leq L/8C\, A_{X,\ell}$, where C and $A_{X,\ell}$ are as in Theorem 3.1.1 and Proposition 2.1.4, respectively. Let us choose N , for t , as in Lemma 3.2.5 and set

$$V_{N,n} = \bigcup_{k=1}^{N} (V_r^{n_k} \cap G_{\mu,n} \cap S_{X,\ell}(x)) .$$

So if $\delta = HD(V_r \cap G_\mu)$ then $HD(V_{N,n}) \le \delta$ for any $n \ge 0$.

Now we choose a fine cover $\mathcal{U}_n$ of $V_{N,n}$ by sets $U \subset V_{N,n}$ such that diam $U \subset L/4$ and

$$\sum_{U \in \mathcal{U}_n} (\text{diam } U)^{\delta+\beta} < 2^{-n} \quad .$$

We define U^* as $\bigcup_{z \in U} D(z,t)$, where $D(z,t)$ denotes the 1-disk of radius t centred at z on $W^s(z)$. For the time being we shall drop A of the notation since it is fixed, so $n(U^*) = n_A(U^*)$ and diam $f^{n(U^*)}U^* \ge L$. By construction it is obvious that diam $U^* < L/2$.

To estimate $n(U^*)$ we take $y_1, y_2 \in U^*$ so that $y_i \in D(z(y_i),t)$ for $z(y_i) \in U$. By Theorem 3.1.1 and the Mean Value Theorem we obtain that, if for each $0 \le j < k$ $d(f^j(z(y_1)), f^j(z(y_2))) < L/4$ then

$$d(f^k(y_1), f^k(y_2)) \le$$

$$d(f^k(y_1), f^k(z(y_1))) + d(f^k(z(y_1)), f^k(z(y_2))) + d(f^k(z(y_2)), f^k(y_2)) \le$$

$$2tCA_{X,\ell} \exp - \frac{99}{100} Xk + d(f^k(z(y_1)), f^k(z(y_2))) \le$$

$$L/4 + d(z(y_1), z(y_2)) \prod_{j=0}^{k-1} (||D_{f^j(z(y_1))} f|| + \alpha) .$$

By Choosing $\mathcal{U}_n$ so fine that $n(U^*) > n$, for each $U \in \mathcal{U}_n$, we have that if $z(y_1) \in U$, $U \in \mathcal{U}_n$, then $z(y_1) \in G_{\mu,n}$ and

$$\prod_{j=0}^{n(U^*)-1} (||D_{f^j(z(y_1))} f||+\alpha) \le \exp(\int \log(||D_z f||+\alpha)d\mu+\beta)n(U^*) .$$

Hence, $n(U^*)$ satisfies

$$L \le \operatorname{diam} f^{n(U^*)}U^* \le L/4+\operatorname{diam} U \exp(\int \log(||D_z f||+\alpha)d\mu+\beta)n(U^*) ,$$

which implies that

$$\exp-(\int \log(||D_z f||+\alpha)d\mu + \beta)n(U^*) \le (3L/4)^{-1}\operatorname{diam} U .$$

Therefore

$$\sum_{U^* \in \mathcal{U}_n^*} \exp -(\delta+\beta)(\int \log(||D_z f||+\alpha)d\mu + \beta)n(U^*) \le$$

$$(3L/4)^{-(\delta+\beta)} \sum_{U \in \mathcal{U}_n} (\operatorname{diam} U)^{\delta+\beta} \le (3L/4)^{-(\delta+\beta)}\ 2^{-n} .$$

Now combining all covers $\mathcal{U}_n^*$ for all $n > q$ such that $n(U^*) > n$ for each $U \in \mathcal{U}_n$, we obtain a cover of $\Lambda_{\bar{x},\ell,\mu} \cap G_\mu \cap \hat{C}(x,h/2)$ with

$$\sum_{n>q} \sum_{U \in \mathcal{U}_n} \exp-(\delta+\beta)(\int \log(||D_z f||+\alpha)d\mu+\beta)n(U^*) \le (3L/4)^{-(\delta+\beta)} .$$

Thus

$$h_A(\Lambda_{\chi,\ell,\mu} \cap G_\mu \cap \hat{C}(x,h/2),f) \leq (\delta+\beta)\left(\int \log(||D_z f||+\alpha)d\mu+\beta\right) ,$$

and since $\beta > 0$ is arbitrary

$$h_A(\Lambda_{\chi,\ell,\mu} \cap G_\mu \cap \hat{C}(x,h/2),f) \leq \delta \int \log(||D_z f||+\alpha)d\mu .$$

Now observe that the above inequality is true for any finite open cover $\mathcal{B}$ with diam $\mathcal{B} < L/2$. Therefore by letting $\alpha \to 0$ it follows that

$$h_\mu(f) \leq h(\Lambda_{\chi,\ell,\mu} \cap G_\mu \cap \hat{C}(x,h/2),f) \leq HD(V_r \cap G_\mu) \int \log||D_z f||d\mu .$$

As in Theorem 1.3.1 we now apply the Ergodic Decomposition Theorem of Entropy [10] and the above procedure to f^n , $n > 0$, to obtain that

$$h_\mu(f^n) \leq HD(V_r \cap G_\mu) \int \log||D_z f^n||d\mu .$$

Since $h_\mu(f^n) = n\, h_\mu(f)$, then by the Subadditive Ergodic Theorem [14] we have

$$h_\mu(f) \leq HD(V_r \cap G_\mu)\chi_\mu^+ . \qquad \square$$

Let μ be a measure as in Theorem 3.2.1. Then for μ-almost every $x \in M$, $W^u(x)$ is an immersed submanifold in M. Thus $W^u(x)$ inherits from M a Riemannian structure and hence a Riemannian measure ν_x, that we shall call the induced measure on $W^u(x)$.

A measure μ, as above, is said to be <u>absolutely continuous on the unstable leaves of</u> f if for any measurable partition ξ of M such that $\xi(x) \subset W^u(x)$ and $\nu_x(\xi(x)) > 0$ for μ-almost every $x \in M$, the conditional measures $\mu_{\xi(x)}$ are absolutely continuous with respect to the induced measure ν_x. See [17] for more details and reference about this definition.

<u>Corollary 3.2.6.</u>

<u>Let</u> $f: M \to M$ <u>be a</u> C^2 <u>diffeomorphism preserving an ergodic Borel probability measure</u> μ. <u>Suppose that</u> μ <u>has non zero exponents of different signs and is absolutely continuous on the unstable leaves of</u> f. <u>Then</u> $HD(G_\mu) = 2$.

<u>Proof</u>.

If μ satisfies the above conditions then by a theorem of Ledrappier and Strelcyn [17] $h_\mu(f) = \chi_\mu^+$. □

<u>Note</u>.

If $\Lambda \subset M$ is an attractor for f supporting a measure μ absolutely continuous on the unstable leaves of f, then Λ is sometimes called a strange attractor. So by the above corollary if Λ is a strange attractor for f, then there exists a measure μ such that $HD(G_\mu) = 2$. Of course the difficult problem is showing the existence of such a measure. □

Now let $x \in \Lambda_{X,\ell,\mu}$ be as in the proof of Theorem 3.2.1, then $W^u_{loc}(x)$ is an admissible $(u,1)$-manifold near x. We define the <u>quotient measure</u> $\tilde{\mu}_x$ on $W^u_{loc}(x)$ given by $S_{X,\ell}(x)$ as follows: if $F \subset W^u_{loc}(x)$ then $\tilde{\mu}_x(F) = \mu(F^s \cap \hat{C}(x,h/2))|\mu(\hat{C}(x,h/2))$, where $F^s = \bigcup_{z(y)\in F} W^s_{loc}(y)$ and $z(y) \in W^s_{loc}(y) \cap W^u_{loc}(x)$.

<u>Corollary 3.2.7</u>.

<u>Let</u> $f:M \to M$ <u>be a</u> C^2 <u>diffeomorphism of a surface</u> M <u>preserving an ergodic Borel probability measure</u> μ. <u>Suppose that</u> μ <u>has non-zero Lyapunov exponents of different signs and that for some</u> $\ell > 1$ $\mu(\Lambda_{X,\ell}) > 0$, <u>where</u> X <u>is the minimum of the absolute values of the Lyapunov exponents of</u> μ. <u>Then for any</u> $x \in \Lambda_{X,\ell,\mu}$

$$h_\mu(f) \le HD(\tilde{\mu}_x)X^+_\mu ,$$

<u>where</u> $\tilde{\mu}_x$ <u>is the quotient measure on</u> $W^u_{loc}(x)$ <u>defined by</u> $S_{X,\ell}(x)$.

<u>Proof</u>.

The idea of the proof is to apply the same argument of part of the proof of Proposition 3.2.4 to a thickened $W^u_{loc}(x)$, for this we need the following statement.

Lemma 3.2.8.

Let $\{A_n\}_{n=1}^{\infty}$ be a collection of finite open covers of M , with diam $A_n \to 0$, and $\{A_n\}_{n=1}$ a collection of sets of positive measure, then

$$h_\mu(f) \le \sup_n h_{A_n}(A_n, f) .$$

Proof.

It is easy to verify that for any finite open cover A and any set $Y \subset M$, $h_A(Y,f) = h_A(fY,f)$ and

$$h_A(\bigcup_{k=0}^{\infty} f^k Y, f) = \sup_k h_A(f^k Y, f) = h_A(Y,f) .$$

For each $n \ge 1$ let $B_n = \bigcup_{k=0}^{\infty} f^k A_n$, then by ergodicity each B_n has measure 1 , and so does $B = \bigcap_{n=1}^{\infty} B_n$. Then for each $n \ge 1$,

$$h_{A_n}(B,f) \le h_{A_n}(B_n,f) = h_{A_n}(A_n,f) .$$

Since diam $A_n \to 0$, as $n \to \infty$

$$h(B,f) = \sup_n h_{A_n}(B,f) \le \sup_n h_{A_n}(A_n,f) ,$$

and by Theorem 1.1.1 $h_\mu(f) \le h(\mathcal{B},f)$, from which the lemma follows. □

To prove the corollary consider a finite open cover A_n of M , and let L_n be a Lebesgue number for A_n . For $0 < t_n \le L_n/8C\,\Lambda_{x,\ell}$, $W^s(W^u_{loc}(x),t_n) \cap \hat{C}(x,h/2)$ has positive measure since $x \in \Lambda_{x,\ell,\mu}$. Now we proceed as in the proof of Proposition 3.2.4 to show that

$$h_{A_n}(W^s(W^u_{loc}(x) \cap \hat{C}(x,h),t_n),f) \le (\delta+\beta)\left(\int \log(||D_zf||+\alpha)d\mu+\beta\right)$$

where $\alpha,\beta > 0$ are exactly the same as the proof of Proposition 3.2.4 and $\delta = HD(W^u_{loc}(x) \cap S_{x,\ell,\mu}(x))$.

Now if $\{A_n\}_{n=1}^\infty$ is a collection of finite open covers of M with diam $A_n \to 0$, then by Lemma 3.2.8, since α,β are arbitrary, it follows that

$$h_\mu(f) \le HD(W^u_{loc}(x) \cap S_{x,\ell,\mu}(x))\int \log||D_zf||d\mu \quad .$$

But let us notice that $S_{x,\ell,\mu}$ could be replaced by any collection of local stable manifolds contained in $S_{x,\ell}(x)$ such that its intersection with $C(x,h/2)$ has measure equal to $\Lambda_{x,\ell} \cap \hat{C}(x,h/2)$. Thus

$$h_\mu(f) \le HD(\tilde{\mu}_x) \int \log||D_zf||d\mu \quad .$$

The rest of this proof is exactly as the ones for Theorem 1.3.1 and Proposition 2.3.4. □

§3. On measures supported on horseshoes.

Let $\tau:\mathrm{Diff}^2(M)\to[0,1]$ be the real valued function on the set $\mathrm{Diff}^2(M)$ of C^2 diffeomorphisms of a surface M defined by

$$\tau(f)=\begin{cases}0 & \text{if } h(f)=0\,,\\ \sup\{h_\mu(f)/\chi_\mu^+ : \mu \text{ is ergodic, } h_\mu(f)>0\} & \text{if } h(f)>0\,.\end{cases}$$

Here we assume that all the measures are Borel probabilities on M and that they are invariant with respect to the diffeomorphisms in question. We recall that $\phi^+=\max\{0,\phi\}$.

Let us take $f\in\mathrm{Diff}^2(M)$, for $n>0$ and $t\in[0,1]$ we write $P(f,n,t)$ for $P(-t\log^+||D_xf^n||^{1/n},f)$. The function $P(f,n,\cdot)$ is continuous and decreasing (not necessarily strictly) on t , with $P(f,n,0)=h(f)$ and $P(f,n,1)\le 0$ by Ruelle's inequality [34] and the fact that if μ is ergodic $\chi_\mu^+=\inf_n\frac{1}{n}\int\log||D_zf^n||d\mu$. Therefore for each $n>0$ there exists $t\in[0,1]$ such that $P(f,n,t)=0$.

Proposition 3.3.1.

For $f\in\mathrm{Diff}^2(M)$:

i) *If* $\chi^+:M\to\mathbb{R}$ *is the measurable function defined by*

$$\chi^+(x)=\limsup_{n\to\infty}\frac{1}{n}\log^+||D_xf^n||\,,$$

then

$$\sup_{\mu \text{ ergodic}} \{h_\mu(f) - \tau(f) \int X^+ d_\mu\} = 0 \ .$$

ii) Let $\tau_n = \inf\{t : P(f,n,t) = 0\}$, then

$$\tau(f) = \sup_n \tau_n \ .$$

Proof.

Let us suppose that $\tau(f) > 0$, otherwise is trivial. By the Subadditive Ergodic Theorem [14] and the Variational Principle [38] it follows that

$$\begin{aligned}
\sup_n P(f,n,t) &= \sup_n \sup_\mu \{h_\mu(f) - \frac{t}{n} \int \log^+ ||D_x f^n|| d\mu\} \\
&= \sup_\mu \sup_n \{h_\mu(f) - \frac{t}{n} \int \log^+ ||D_x f^n|| d\mu\} \\
&= \sup_\mu \{h_\mu(f) - t \int X^+ d\mu\} \quad .
\end{aligned}$$

Hence

$$\sup_n P(f,n,\sup_n \tau_n) = \sup_\mu \{h_\mu(f) - \sup_n \tau_n \int X^+ d\mu\} \quad ,$$

by the definition of τ_n $P(f,n,\sup_n \tau_n) \le 0$, so

$$\sup_\mu\{h_\mu(f) - \sup_n \tau_n \int X^+ d\mu\} \quad .$$

On the other hand we have that

$$0 = \sup_n P(f,n,\tau_n) = \sup_n \sup_\mu\{h_\mu(f) - \frac{\tau_n}{n}\int \log^+||D_x f^n||d\mu\}$$

$$\le \sup_\mu\{h_\mu(f) + \sup_n \tau_n \sup_n - \frac{1}{n}\int \log^+||D_x f^n||d\mu\}$$

$$= \sup_\mu\{h_\mu(f) - \sup_n \tau_n \int X^+ d\mu\} \quad .$$

(By the Subadditive Ergodic Theorem $\int X^+ d\mu = \inf_n \frac{1}{n}\int \log||D_x f^n||d\mu$) .

Thus

$$\sup_\mu\{h_\mu(f) - \sup_n \tau_n \int X^+ d\mu\} = 0 \ .$$

Since $X_\mu^+ = \int X^+ d\mu$ when μ is ergodic and $h_\mu(f) \le X_\mu^+$ by Ruelle's inequality, then if $h_\mu(f) > 0$

$$h_\mu(f) - \sup_n \tau_n \int X^+ d\mu \le 0$$

implies that $h_\mu(f)/X_\mu^+ - \sup_n \tau_n \le 0$, and therefore $\tau(f) \le \sup_n \tau_n$.

Now suppose the above inequality is strict, then $P(f,n,\tau(f)) > 0$ for some $n > 0$, so there exists ν ergodic with $h_\nu(f) > 0$ such that

$$h_\nu(f) - \tau(f)\int \log^+||D_x f^n||^{1/n} d\nu > 0 .$$

But since $0 < \chi_\nu^+ \le \int \log||D_x f^n||^{1/n} d\nu$ then

$$h_\nu(f) \le h_\nu(f) \int \log^+||D_x f^n||^{1/n} d\nu / \chi_\nu^+ ,$$

and $h_\nu(f)/\chi_\nu^+ - \tau(f) > 0$, a contradiction. So we have proved i) and ii). □

Theorem 3.3.2.

Let $f \in \mathrm{Diff}^2(M)$ with M a surface. If $\tau(f) > 0$ then

$$\tau(f) = \sup\{\tau(f|\Lambda) : \Lambda \text{ is hyperbolic}\} .$$

Moreover, the sup can be taken over horseshoes.

Proof.

It is sufficient to show that for $0 \le t < \tau(f)$

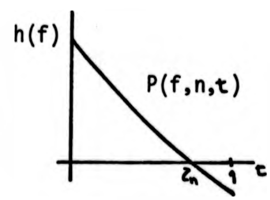
$$\sup_{\mu}\{h_{\mu}(f)-t\int x^{+}d\mu\} = \sup_{\substack{\mu \\ \text{supp } \mu \text{ is hyperbolic}}} \{h_{\mu}(f)-t\int x^{+}d\mu\} \quad .$$

Here we are assuming that all the measures are ergodic.

We recall that

$$\sup_{\mu}\{h_{\mu}(f)-t\int x^{+}d\mu\} = \sup_{n} P(f,n,t) \ ,$$

then if $t < \tau_n$ for some large n , $P(f,n,t) > 0$.



Therefore by Theorem 2.3.2 for n large

$$P(-t/n \log^{+}||D_x f^n||,f) = \sup_{\Lambda \text{ hyperbolic}} P(-t/n \log^{+}||D_x f^n|||\Lambda,f|\Lambda) \ ,$$

since $-t/n \log^{+}||D_x f^n||$ is continuous on x and if $t < \tau_n$ then we can take the sup over measures with nonzero exponents in

$$\sup_{\mu}\{h_{\mu}(f) - t/n\int \log^{+}||D_x f^n||d\mu\} \quad .$$

And it follows from this last paragraph that

$$\tau(f) = \sup\{\tau(f|\Lambda) \;|\Lambda \text{ is hyperbolic}\} .$$

□

<u>Note</u>.

When Λ is a hyperbolic set $\tau(f|\Lambda)$ is the Hausdorff dimension of the intersection of the unstable manifold of a point $x \in \Lambda$ with Λ , see [19] .

Now we endow $\mathrm{Diff}^2(M)$ with the C^2 topology [25].

<u>Corollary 3.3.3</u>.

<u>The function</u> $\tau:\mathrm{Diff}^2(M) \to [0,1]$, <u>with</u> M <u>a surface</u>, <u>is lower semicontinuous</u>.

<u>Proof</u>.

By the result of McCluskey and Manning [19] given $\Lambda \subset M$ hyperbolic for f and $\varepsilon > 0$ there exists a neighbourhood N of f in $\mathrm{Diff}^2(M)$, such that for every $g \in N$ there exists a set $\Lambda_g \subset M$ hyperbolic for g with $\tau(f|\Lambda) \leq \tau(f|\Lambda_g) + \varepsilon/2$.

Now if Λ was chosen such that $\tau(f) < \tau(f|\Lambda) + \varepsilon/2$, then for any $g \in N$, $\tau(f) \leq \tau(g) + \varepsilon$. Which proves that τ is lower semicontinuous.

□

Corollary 3.3.4.

If $\tau(f) > 0$, *then for any* $\delta > 0$ *there exists an ergodic f-invariant measure* μ *supported on a horseshoe such that*

$$HD(G_\mu) \geq 1 + \tau(f) - \delta \quad ,$$

furthermore the $1 + h_\mu(f)/X_\mu^+$ *Hausdorff measure of* G_μ *is finite and positive.*

Proof.

Given $\delta > 0$ we use Theorem 3.3.2 to choose a horseshoe Λ such that $\tau(f) \leq \tau(f|\Lambda) + \delta$. Now we choose the generalized Bowen-Ruelle measure μ for Λ , see [9] and [19], that is a measure μ on Λ such that $\tau(f|\Lambda) = h_\mu(f)/X_\mu^+$. Then by Theorem 3.2.1

$$HD(G_\mu) \geq 1 + h_\mu(f)/X_\mu^+ \geq 1 + \tau(f) - \delta \quad .$$

The $1 + h_\mu(f)/X_\mu^+$ Hausdorff measure of G_μ is finite and positive by Lemma 10 of [4] and the fact that the family of local stable manifolds of a hyperbolic set of a C^2 diffeomorphism of a surface is Lipschitz [12]. □

Note.

We should notice that a combination of Lemma 10 of [4] and the

Lipschitz condition on the family of local stable manifolds, leads to a shorter proof of Ruelle's Theorem [33] for attractors on surfaces.

□

Corollary 3.3.5.

Let $f \in Diff^2(M)$ with M a surface. If $P(-X^+) = \sup_\mu\{h_\mu(f)-\int X^+ d\mu\}$ and $h(f) > 0$, then

$$\tau(f) \leq h(f)/(h(f)-P(-X^+)) .$$

Proof.

By the Variational Principle [38]

$$P(f,n,t) = \sup_\mu\{h_\mu(f) - t/n \int \log^+ \|D_x f^n\| d\mu\}$$

$$= \sup_\mu\{(h_\mu(f) - th_\mu(f))+(th_\mu(f)-t/n\int \log^+\|D_x f^n\| d\mu)\}$$

$$\leq (1-t)h(f) + tP(f,n,1) .$$

Now we take $\sup_n$ to obtain that

$$\sup_\mu\{h_\mu(f)-t\int X^+ d\mu\} \leq (1-t)h(f) + tP(-X^+) ,$$

from where it follows that

$$\tau(f) \leq h(f) / (h(f) - P(-X^+)) . \qquad \square$$

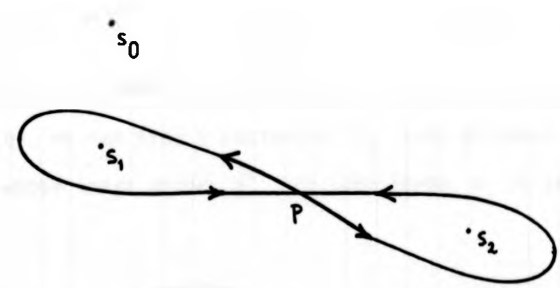
<u>Theorem 3.3.5.</u>

<u>The function</u> $\tau:\mathrm{Diff}^2(M) \to [0,1]$, <u>where</u> M <u>is a surface, is not continuous</u>.

We shall only give an outline of the proof, since it is basically contained in [24], to which we refer for details.

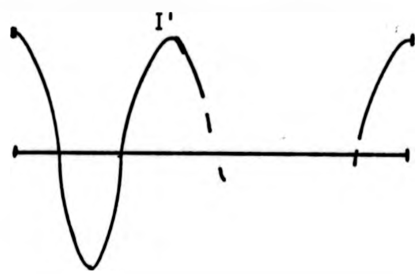
<u>Outline of the proof</u>.

Let $f:S^2 \to S^2$ be a C^2 diffeomorphism of the 2-sphere S^2 with three fixed sources (expanding fixed points) s_0, s_1, s_2 and a hyperbolic fixed point p of saddle type. Suppose that $W^s(p) = W^u(p)$, then they form a figure 8 curve that divides S^2 into three regions. In each region there is a source and any other point is attracted to the figure 8 curve.

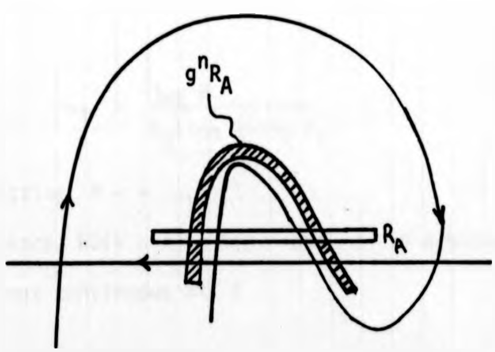


We may assume that f is a C^2 diffeomorphism of $\mathbb{R}^2$ with so a source at infinity and an interval I of $W^s(p)\setminus\{p\}$ lying in the x-axis. Suppose that $I = [-a,a]$, then we can introduce bumps on

$W^u(p)$ over I without disturbing the fact that $I \subset W^s(p)$. Let I' denote the piece of $W^u(p)$ over I , we arrange I' to be the graph of the function $x \to A \cos \pi Nx/2a$ with $A = K_1(\pi N/2)^{-1}\delta$, N a large positive integer, $\delta > 0$ and K_1 constants independent of f . If we call the perturbed diffeomorphism g , then g is δ-C^2 close to f .



Let us denote by $W^u(p,g)(W^s(p,g))$ the unstable (stable) manifold of p , that we suppose to remain fixed. Since $I \subset W^s(p,q)$ and $I' \subset W^u(p,q)$ we can find a rectangle R_A with distance around $A/4$ from I , whose image under g^n (or some large n is around $A/4$ of I' .



The set $g^n R_A \cap R_A$ has N components and for $k \in \mathbb{Z}$, $g^{nk} R_A \cap R_A$ has N^k components. So we have a horseshoe Λ of N folds and period n.

Now we take a piece of unstable manifold of a point in the horseshoe, denote it by V, such that it only intersects R_A once. Then $V \cap \Lambda$ can be covered by N^k subsets of V of diameters larger than $(K_2|\lambda(p)|^{-n})^k$ where $\lambda(p)$ is the largest eigenvalue of $D_p g$, K_2 is a constant independent of N and $k \geq 0$.

The Hausdorff dimension of $V \cap \Lambda$, $HD(V \cap \Lambda)$, satisfies

$$HD(V \cap \Lambda) \geq \alpha_1 = \inf\{\beta : \inf_{k \geq 0} N^k (K_2|\lambda(p)|^{-n})^{k\beta}) = 0\} .$$

Then α_1 is given by $N(K_2|\lambda(p)|^{-n}|^{\alpha_1} = 1$, so

$$\alpha_1 = \frac{\log N}{n \log|\lambda(p)| - \log K_2} .$$

But for some constant K_3 independent of N,

$$n \log|\lambda(p)| < K_3 + \log N ,$$

so

$$\alpha_1 > \frac{\log N}{K_3 + \log N - \log K_1} ,$$

and letting $N \to \infty$ $\alpha_1 \to 1$.

Since $HD(V \cap \Lambda) \leq \tau(g)$ and δ is arbitrary, it follows that τ is not continuous at f. $\square$

CHAPTER 4.

A Relation Between Lyapunov Exponents, Hausdorff Dimension and Entropy.

§0. Introduction.

In [18] Manning proved that for an Axiom A diffeomorphism f of a surface M, which preserves an ergodic non-atomic Borel probability measure μ, its entropy $h_\mu(f)$ satisfies

$$h_\mu(f) = \delta_\mu \chi_\mu^+ ,$$

where δ_μ is the Hausdorff dimension of the intersection of the set G_μ of generic points of μ with an interval on the unstable manifold of any point x belonging to the basic set Λ that supports the measure μ, and χ_μ^+ is the positive Lyapunov exponent for μ.

The number δ_μ can be reinterpreted as the Hausdorff dimension of the quotient measure $\tilde{\mu}_x$ given by the family of local stable manifolds through a neighbourhood of x, with $x \in \operatorname{supp} \mu$.

In §2 of this chapter we shall prove that if we ask the family of local stable manifolds $S_{\chi,\ell}(x)$ to be Lipschitz, then using ideas of Mañé [20] and Young [39] we can extend Manning's result to any C^2 diffeomorphism of a surface.

In order to do this we need a slight variation of Mañé's lower bound of entropy [20] and a local approach to Hausdorff dimension of a measure [2], [39] .

§1. Mañé's Lower Bound For Entropy.

Let $f:M \to M$ be a continuous map of a compact finite dimensional manifold M. If $\psi:M \to (0,1)$ is a function, then for $x \in M$ and $n \geq 0$ define

$$B(x,\psi,n) = \{y \in M : d(f^i(x),f^i(y)) \leq \psi(f^i(x)),\ 0 \leq i \leq n\} \quad .$$

Proposition 4.1.1.

Let $f:M \to M$ *be as above, and suppose that* μ *is an ergodic* f-*invariant Borel probability measure. Let* $Y \subset M$ *be a set of positive* μ-*measure and denote by* μ_Y *the conditional measure on* Y , *then*

$$\limsup_{n\to\infty} -\frac{1}{n} \log \mu_Y(B(x,\psi,n)) \leq h_\mu(f)$$

for almost every $x \in Y$ *and every* $\psi:M \to (0,1)$ *such that* $\log \psi$ *is* μ *integrable*.

Remark.

The above proposition follows from the proof of a similar statement due to Mañé [20]. Since the case $\mu_Y \ll \mu$ is not covered by Mañé's statement, we give a proof of Proposition 4.1.1 for the sake of completeness.

Proof.

The main part of the proof of Mañé's Proposition is the following lemma.

Lemma 4.1.2. [20]

If μ is a probability measure on M and $\psi : M \to (0,1)$ is such that $\log \psi$ is μ-integrable, then there exists a countable partition P of M with $H(P) < \infty$ such that if $P(x)$ denotes the atom of containing x, then

$$\operatorname{diam} P(x) \leq \psi(x)$$

for almost every $x \in M$. □

Now given $\psi : M \to (0,1)$ μ-integrable take P as in the above lemma. The S.M.B. theorem [28] says that for almost every $x \in M$

$$\lim -\frac{1}{n} \log \mu(P_n(x)) \leq h_\mu(f) .$$

Let P_∞ be the σ-algebra generated by the partitions P_n, $n \geq 0$. By the Radom-Nikodym theorem and the Increasing Martingales theorem [28] there exists $k : M \to \mathbb{R}$, μ_Y-integrable and measurable with respect to the σ-algebra P_∞ such that for all $A \in P_\infty$

$$\int_A k \, d\mu = \mu_Y(A)$$

and for μ-almost every $x \in M$

$$\lim_{n\to\infty} \frac{\mu_Y(P_n(x))}{\mu(P_n(x))} = k(x) .$$

Thus

$$\frac{1}{n} \log \mu_Y(P_n(x)) = \frac{1}{n} \log \mu(P_n(x)) + \frac{1}{n} \log \frac{\mu_Y(P_n(x))}{\mu(P_n(x))} ,$$

and for μ_Y-almost every $x \in M$

$$\lim_{n\to\infty} - \frac{1}{n} \log \mu_Y(P_n(x)) = \lim_{n\to\infty} - \frac{1}{n} \log \mu(P_n(x)) .$$

By Lemma 4.1.2 $P_n(x) \subset B(x,\psi,n)$, therefore

$$\lim_{n\to\infty} - \frac{1}{n} \log \mu_Y(B(x,\psi,n)) \le \lim_{n\to\infty} - \frac{1}{n} \log \mu(P_n(x)) \le h_\mu(f)$$

for almost every $x \in Y$. □

Now we shall take a local approach to Hausdorff dimension. This is basically due to Billingsley [2] and Young [39].

Proposition 4.1.3. [39]

Let μ *be a Borel probability measure on* $\mathbb{R}^n$ *and* $\Lambda \subset \mathbb{R}^n$ *be a* measurable set of positive measure. Suppose that for every $x \in \Lambda$

$$\limsup_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \leq \delta ,$$

then $HD(\Lambda) \leq \delta$. □

§2. Entropy of certain diffeomorphisms of surfaces.

This section is an attempt to generalize Manning's theorem [18] to C^2 diffeomorphisms of surfaces. We shall look at the set of C^2 diffeomorphisms for which the families of local stable manifolds are Lipschitz, this set certainly includes all C^2 Axiom A diffeomorphisms. Pugh and Shub have announced some results that could imply that this is the case for any C^2 diffeomorphisms of a surface, but we have not seen any written version of this work. Now we shall explain what we mean by the family of local stable manifolds being Lipschitz.

Let $f:M \to M$ be a C^2 diffeomorphism of a surface M , preserving an ergodic Borel probability measure μ with nonzero exponents of different signs. Let $\chi = \min\{-\chi_1^{\mu}, \chi_2^{\mu}\}$ and choose $\ell > 1$ such that $\mu(\Lambda_{\chi,\ell}) > 0$. Consider $x \in \Lambda_{\chi,\ell,\mu}$ and let $S_{\chi,\ell}(x)$ denote the family of local stable manifolds through $\hat{C}(x,h/2) \cap \Lambda_{\chi,\ell}$, with $h = h(\chi,\ell)$ of §1 Chapter 3.

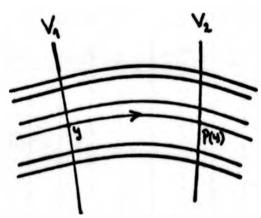
Let us set

$$\Lambda_{X,\ell}(x) = \bigcup_{y \in \Lambda_{X,\ell} \cap C\ell \hat{C}(x,h/2)} C\ell\, W^s_{loc}(y) \cap C\ell\, \hat{C}(x,h/2) \quad .$$

Suppose V^1 and V^2 are two C^1 submanifolds transverse to $S_{X,\ell}(x)$, so that they intersect each local stable manifold only once. Then there exist sets $\tilde{V}^1 \subset V^1$ and $\tilde{V}^2 \subset V^2$ for which the Poincaré map

$$p : \Lambda_{X,\ell}(x) \cap \tilde{V}^1 \to \Lambda_{X,\ell}(x) \cap \tilde{V}^2 ,$$

defined by $p(y) = \tilde{V}^2 \cap W^s_{loc}(w)$ with $y \in \tilde{V}^1 \cap W^s_{loc}(w)$ and $w \in \Lambda_{X,\ell} \cap \hat{C}(x,h)$, is a homeomorphism.



We say that <u>the family of local stable manifolds</u> $S_{X,\ell}(x)$ <u>is Lipschitz</u> if each Poincaré map between any two C^1 submanifolds transverse to $S_{X,\ell}(x)$, constructed as above, is Lipschitz, with Lipschitz constant depending linearly on the C^1 distance between the two submanifolds. See Theorem 3.2.1 of [29] for the motivation of this definition.

Therefore there exists a constant $K = K(\chi,\ell)$ such that if V^1 and V^2 are two admissible $(u,1)$-manifolds near x, $x \in \Lambda_{\chi,\ell}$, then the Lipschitz constant of the Poincaré map between these two submanifolds is less than or equal to K.

We recall that $S_{\chi,\ell}(x) \cap \hat{C}(x,h/2)$ defines a quotient measure $\tilde{\mu}_x$ on $W^u_{loc}(x)$, see Corollary 3.2.7. Similarly define $\tilde{\mu}_V$ for any C^1 submanifold V transverse to $S_{\chi,\ell}(x)$. Thus if the family of local stable manifolds $S_{\chi,\ell}(x)$ is Lipschitz, then $HD(\tilde{\mu}_x) = HD(\tilde{\mu}_V)$.

Theorem 4.2.1.

Let $f:M \to M$ be a C^2 diffeomorphism of a surface M and suppose that μ,χ,ℓ are as above. Let $x \in M$ be a μ-density point for $\Lambda_{\chi,\ell}$ and let V be a C^1 submanifold transverse to $S_{\chi,\ell}(x)$. If the family of local stable manifolds $S_{\chi,\ell}(x)$ is Lipschitz, then

$$h_\mu(f) = HD(\tilde{\mu}_V)\chi^+_\mu ,$$

where χ^+_μ is the positive Lyapunov exponent of μ and $\tilde{\mu}_V$ is the quotient measure on V defined by $S_{\chi,\ell}(x)$.

Proof.

It is sufficient to prove that $h_\mu(f) = HD(\tilde{\mu}_x)\chi^+_\mu$. By Corollary 3.2.7 $h_\mu(f) \le HD(\tilde{\mu}_x)\chi^+_\mu$, so we just need to prove that $HD(\tilde{\mu}_x) \le h_\mu(f)/\chi^+_\mu$. In order to do so we shall follow very closely Mañé's proof of Pesin's formula [20].

Fix $\alpha > 0$ so small that $\mu(\Lambda_{\chi,\ell}) > 2\sqrt{\alpha}$. By Egorov's theorem exists a compact set $\Lambda^1 \subset \Lambda_\chi$, with $\mu(\Lambda^1) \geq 1-\alpha$, such that $T_xM = E_x^s \oplus E_x^u$ varies continuously on Λ^1 and for some $N > 0$, if $g = f^N$, the inequalities

$$||D_x g^n v|| \geq \exp nN(\chi_\mu^+ - \alpha)||v||$$

$$||D_x g^n v|| \leq 1$$

hold for all $x \in \Lambda^1$, $n \geq 0$ and $v \in E_x^u$.

Observe that the Ergodic Theorem implies that

$$\mu(\{x | \lim_{n\to\infty} 1/n \,\#\, \{0 \leq j < n | g^j(x) \notin \Lambda^1\} \leq \sqrt{\alpha}\}) \geq 1 - \sqrt{\alpha} \quad .$$

Then, applying Egorov's theorem once more, there exists a compact set $\Lambda^2 \subset \Lambda^1$ with $\mu(\Lambda^2) \geq 1 - 2\sqrt{\alpha}$ and $N_0 > 0$ such that

$$\#\{0 \leq j < n \mid g^j(x) \notin \Lambda^1\} \leq 2n\sqrt{\alpha}$$

for all $x \in \Lambda^2$. Now let $\Lambda^3 = \Lambda^2 \cap \Lambda_{\chi,\ell}$, clearly $\mu(\Lambda^3) \geq \mu(\Lambda_{\chi,\ell}) - 2\sqrt{\alpha} > 0$.

Let us choose $x \in M$ such that x is a μ-density point for Λ^3 . We define a measure $\tilde{\nu}_x$ on $W^u_{loc}(x)$ as follows: if $A \subset W^u_{loc}(x) \cap \hat{C}(x,h/2)$, then

$$\tilde{\nu}_x(A) = \mu((\bigcup_{z(y)\in A} W^s_{loc}(y) \cap \hat{C}(x,h/2) \cap \Lambda^3)/\mu(\hat{C}(x,h/2) \cap \Lambda^3) \ ,$$

where $z(y) \in W^s_{loc}(y) \cap W^u_{loc}(x)$ and $y \in \hat{C}(x,h/2) \cap \Lambda_{x,\ell}$.

Let us denote by ν the μ-conditional measure on $\hat{C}(x,h/2) \cap \Lambda^3$ and let $B(y,\psi,n) = \{w \in M \mid d(g^i(w),g^i(y)) \le \psi^i(g(y))$ for $0 \le i \le n\}$, where $\psi: M \to (0,1)$ is a function. Then since f is ergodic it follows that for almost every $y \in \hat{C}(x,h/2) \cap \Lambda^3$

$$\limsup_{n\to\infty} -\frac{1}{n} \log \nu(B(y,\psi,n)) \le h_\mu(g) ,$$

provided that $\log \psi$ is μ-integrable.

Now we are going to define the ψ to be used in this proof. For this we need some lemmas from [20].

If $E = E_1 \oplus E_2$ is a normed space, we say that $G \subset E$ is an (E_1,E_2)-graph if there exists an open subset $U \subset E_2$ and a C^1 map $\phi: U \to E_1$ satisfying $G = \{(\phi(v),v) : v \in U\}$. The number $\sup\{\|\phi(v_1)-\phi(v_2)\|/\|v_1-v_2\| \mid v_1,v_2 \in U\}$ is called the dispersion of G .

Lemma 4.2.2. [20]

For all $c > 0$ there exists $\xi > 0$ such that if $y \in \Lambda^1$ and for $m > 0$, $g^m(y) \in \Lambda^1$, then if W is a C^1 submanifold of M such that $\exp_y^{-1} W$ is an (E^s_y,E^u_y)-graph with dispersion $\le c$ and

$W \subset B(x,\xi^m)$, *then* $\exp^{-1}_{g^m(y)} g^m W$ *is a* $(D_y g^m E^s_y, D_y g^m E^u_y)$*-graph with dispersion* $\leq c$. □

Remark 4.2.3.

For each $y \in \Lambda_{x,\ell}$, $W^u_{loc}(y)$ can be lifted through the exponential map as an (E^s_y, E^u_y)-graph. Furthermore, as in Remark 2.3.1 of [29], for any small $c > 0$ we can find $\delta > 0$ such that if $z \in B(y,\delta) \cap \Lambda_{x,\ell}$, then $W^u_{loc}(z) \cap B(y,\delta)$ can be lifted through the exponential map as an (E^s_y, E^u_y)-graph with dispersion $\leq c$.

Lemma 4.2.4. [20] [29]

For $\alpha > 0$ *there exist* $a > 0$ *and* $c > 0$ *such that if* $y \in \Lambda^1$, $z \in M$ *and* $d(z,y) < a$, *then for any* C^1 *submanifold of* M *such that* $z \in W$ *and* $\exp_y^{-1} W$ *is an* (E^s_y, E^u_y)*-graph with dispersion* $\leq c$ *we have*

$$\left| \log \| D_z g | T_z W \| - \log \| D_y g | E^u_y \| \right| < \alpha \quad .$$ □

Choose c, a, ξ and δ by the above lemmas and Remark 4.2.3.

Let

$$r(y) = \begin{cases} 0 & \text{if } y \notin \Lambda^1 \\ \text{the smallest integer } k > 0 \text{ such that } g^k(y) \in \Lambda^1 & \text{if } y \in \Lambda_1 \, , \end{cases}$$

define $\psi: M \to (0,1)$ by

$$\psi(y) = \min\{a, \delta, \xi^{r(y)}, \exp(N(\chi_\mu^+ + 2\alpha))^{-r(y)}/\sqrt{2}\} .$$

Since $\int r(y)d\mu \le 1$, then $\log \psi$ is μ-integrable.

For small $\rho > 0$ define $n(\rho)$ to be the smallest positive integer satisfying

$$\rho \exp\{n(\rho)[N(\chi_\mu^+ - \alpha) - \alpha - 4NC\sqrt{\alpha}]\} \ge 1 ,$$

where $C = \max\{\sup_{y \in M} \log||D_y f||, \sup_{y \in M} \log||D_y f^{-1}||\}$. It is obvious that for small ρ, $2n(\rho)\sqrt{\alpha} > N_0$.

The proof of the following lemma is essentially borrowed from [20] and [39].

Lemma 4.2.5.

For sufficiently small $\rho > 0$ and any $y, w \in \Lambda^3 \cap \hat{C}(x,h)$ such that $w \in B(y, \psi, n(\rho))$, then

$$W^u_{loc}(w) \cap B(y, \psi, n(\rho)) \subset B(z(w), \rho) \cap W^u_{loc}(z(w)) ,$$

where $z(w) \in W^u_{loc}(w) \cap W^s_{loc}(y)$.

Proof.

For any $y \in \Lambda^3 \cap \hat{C}(x,h)$ let $\{n_0,n_1,\ldots\} = \{n \geq 0 : g^n(y) \in \Lambda^1\}$, assume that $n_0 < n_1 < \ldots < n_k < n(\rho) < n_{k+1} < \ldots\ldots$. Let us write $W^u(w,\psi,n(\rho))$ for $W^u_{loc}(w) \cap B(y,\psi,n(\rho))$, for any $w \in B(y,\psi,n(\rho)) \cap \Lambda^3 \cap \hat{C}(x,h/2)$.

So for small $\rho > 0$ then $n_k \geq N_0$, since $2n(\rho)\sqrt{\alpha} > N_0$ and $y \in \Lambda^2$. Denote $\{0 \leq n_i \leq n\}$ by S_n . Thus if $n_i \in S_{n_k}$ by Lemma 4.2.2 $g^{n_i}(W^u(w,\psi,n(\rho))$ can be lifted as an $(E^s_{g^{n_i}(y)}, E^u_{g^{n_i}(y)})$-graph with dispersion $\leq c$ (since from the definition of ψ ,

$$g^{n_{i-j}}W^u(w,\psi,n(\rho)) \subset B(g^{n_{i-j}}(y),\varepsilon^{n_i-n_{i-j}}) \text{ for all } 0 \leq j < i \text{).}$$

And by Lemma 4.2.4 if $n_i \in S_{n_k}$

$$|\log||D_{g^{n_i}(z')}g|T_{g^{n_i}(z')}g^{n_i}W^u(w,\psi,n(\rho))||-\log||D_{g^{n_i}(y)}g|E^u_{g^{n_i}(y)}||| < \alpha ,$$

for any $z' \in W^u(w,\psi,n(\rho))$, also by the definition of ψ .

Therefore, since $W^u(w,\psi,n(\rho))$ is one dimensional

$$\log||D_{z'}g^{n_k}|T_{z'}W^u(w,\psi,n(\rho))|| = \sum_{i=0}^{n_k-1}\log||D_{g^i(z')}g|T_{g^i(z)}g^iW^u(w,\psi,n(\rho))||$$

$$\geq \sum_{i\in S_{n_k}}\log||D_{g^i(z')}g|T_{g^i(z')}g^iW^u(w,\psi,n(\rho))||-(n_k-\#S_{n_k})NC$$

$$\geq \sum_{i\in S_{n_k}} \log||D_{g^i(y)} g|E^u_{g^i(y)}|| - \alpha n_k - (n_k - \# S_{n_k})NC$$

$$\geq \sum_{i=1}^{n_k-1} \log||D_{g^i(y)} g|E^u_{g^i(y)}|| - \alpha n_k - 2(n_k - \# S_{n_k})NC$$

$$= \log||D_y g^{n_k}|E^u_y|| - \alpha n_k - 4n_k NC\sqrt{\alpha}$$

$$\geq n_k N(\chi^+_\mu - \alpha) - \alpha n_k - 4n_k NC\sqrt{\alpha}$$

$$= n_k(N(\chi^+_\mu - \alpha) - \alpha - 4NC\sqrt{\alpha}) \quad .$$

Thus

$$(*) \quad ||D_{z'} g^{n_k}|T_{z'} W^u(w,\psi,n(\rho))|| \geq \exp n_k(N(\chi^+_\mu - \alpha) - \alpha - 4NC\sqrt{\alpha})$$

for any $z' \in W^u(w,\psi,n(\rho))$.

Now let $d_i'(\cdot,\cdot)$ denote the restriction of the Riemannian metric to $g^i W^u(w,\psi,n(\rho))$. Since $g^{n_k} W^u(w,\psi,n(\rho))$ can be lifted as an $(E^s_{g^{n_k}(y)}, E^u_{g^{n_k}(y)})$-graph with dispersion $\leq c$, and c is small, it follows that $d_{n_k}'(\cdot,\cdot) \leq \sqrt{2}\, d(\cdot,\cdot)$. Obviously $d(\cdot,\cdot) \leq d_0'(\cdot,\cdot)$.

By the Mean Value Theorem if $w' \in W^u(w,\psi,n(\rho))$, then

$$d'_{n_k}(g^{n_k}(w'),g^{n_k}(z(w))) = ||D_{z'}g^{n_k}|T_{z'}W^u(w,\psi,n(\rho))||d'_0(w',z(w))$$

for some $z' \in W^u(w,\psi,n(\rho))$. Therefore, by (*) and the definition of $n(\rho)$,

$$d'_{n_k}(g^{n_k}(w'),g^{n_k}(z(w)) \geq \exp n_k(N(\chi_\mu^+-\alpha)-\alpha-4NC\sqrt{\alpha})d'_0(w',z(w))$$

$$\geq \exp(-n(\rho)+n_k)(N(\chi_\mu^+-\alpha)-\alpha-4NC\sqrt{\alpha})\frac{d'_0(w',z(w))}{\rho} .$$

Now since $r(g^{n_k}(y)) = n_{k-1}-n_k$ and

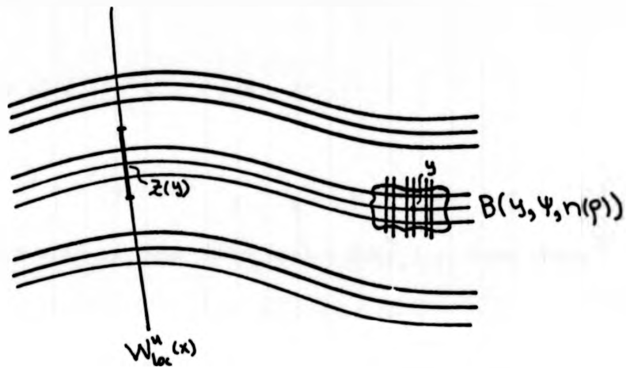
$$d(g^{n_k}(w'),g^{n_k}(z(w))) \leq \psi(g^{n_k}(y)) \leq \exp(-n_{k+1}+n_k)(N(\chi_\mu^+-\alpha)-\alpha-4NC\sqrt{\alpha})/\sqrt{2} ,$$

it follows that $d(w',z(w)) \leq \rho$ and the lemma is proved. □

Continuing with the main argument, let us note that the last lemma implies that

$$\nu(B(y,\psi,n(\rho)) \leq \nu(\bigcup_{w\in B(y,\psi,n(\rho))\cap\Lambda^3} W^u(w,\psi,n(\rho)))$$

$$\leq \nu(\bigcup_{w\in B(y,\psi,n(\rho))\cap\Lambda^3} B(z(w),\rho) \cap W^u_{loc}(w)) .$$



The Lipschitz property of $S_{X,\ell}(x)$ implies that the projection along $S_{X,\ell}(x)$ of $\bigcup_{w \in B(y,\psi,n(\rho)) \cap \Lambda^3} B(z(w),\rho) \cap W^u_{loc}(w)$ is contained in $B(z(y),K\rho) \cap W^u_{loc}(x)$, when $z(y) \in W^u_{loc}(x) \cap W^s_{loc}(y)$.

Thus, by the definition of $\tilde{\nu}_x$

$$\nu(B(y,\psi,n(\rho)) \le \tilde{\nu}_x(B(z(y),K\rho)) .$$

Therefore

$$-\frac{1}{n(\rho)} \log\nu(B(y,\psi,n(\rho)) \ge (N(X_\mu^+-\alpha)-\alpha-4CN\sqrt{\alpha}) \frac{\log\tilde{\nu}_x(B(z(y)),K\rho))}{\log \rho}$$

and by Propositions 4.1.1 and 4.1.3 it follows that

$$h_\mu(f) \geq HD(\tilde{\nu}_X)(\chi_\mu^+ - \alpha - \alpha/N - 4C\sqrt{\alpha}) .$$

Now observe that $\Lambda^3 = \Lambda^3(\alpha)$ and $\tilde{\nu}_X = \tilde{\nu}_X(\alpha)$, and as $\alpha \to 0$ we have $\mu(\Lambda^3(\alpha)) \to \mu(\Lambda_{X,\ell})$ and $HD(\tilde{\nu}_X(\alpha)) \to HD(\tilde{\mu}_X)$. From where it follows that

$$h_\mu(f) \geq HD(\tilde{\mu}_X)\chi_\mu^+ \quad . \qquad \square$$

CHAPTER 5.

Topological Entropy of Homoclinic Closures.

§0. Introduction.

In this chapter we shall study the topological entropy of certain invariant sets of diffeomorphisms of surfaces. In order to do so we shall combine ideas of Bowen [5], [6] and Katok [13].

The motivation behind this work is the following. If $f:M \to M$ is a C^1 Axiom A diffeomorphism of a manifold M, the non-wandering set of f, $\Omega(f)$, can be written as the union of a finite number of closed f-invariant disjoint sets, say $\Omega(f) = \Lambda_1 \cup .. \cup \Lambda_t$. Moreover, for each Λ_i, $1 \le i \le t$, there exists a point $x \in \Lambda_i$ such that the orbit of x is dense in Λ_i. The topological entropy $h(f) = \sup_i h(\Lambda_i, f)$ and by Bowen [5]

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log \# \{x \in M : f^n(x) = x\} \quad .$$

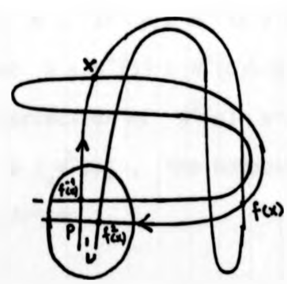
It also follows from Bowen's work that $h(\Lambda_i, f) > 0$ iff for any periodic point $p \in \Lambda_i$ the unstable manifold of p, $W^u(p)$, intersects the stable manifold of p, $W^s(p)$. Furthermore,

$$\Lambda_i = C\ell\{x \in M \mid x \in W^u(\theta(p)) \cap W^s(\theta(p))\} \quad .$$

A point $x \in W^u(p) \cap W^s(p)\setminus\{p\}$ is called a homoclinic point of p, in this case the hyperbolicity of Λ_i implies that $W^u(p)$ and $W^s(p)$ intersect transversally at x.

So the questions follow, can you talk about growth rates of homoclinic points as you do for periodic points? If so, is there a quantitative relation with topological entropy?

The answers are yes to both questions and very simple if we follow Bowen's proof of the periodic points formula in [6]. But for this we need a way of counting the different orbits of homoclinic points according to the dynamics of f. The idea is as follows, since $x \in W^s(p) \cap W^u(p)$, suppose p is fixed, then $f^n(x) \to p$ as $n \to \pm\infty$. Hence, if $x \notin B(p,\varepsilon)$, x will enter the local stable manifold $W^s_\varepsilon(p)$, for some small $\varepsilon > 0$, in some time n and will never leave it again. Similarly, there is $m \geq 0$ such that for any $k \geq m$, $f^{-k}(x) \in W^u_\varepsilon(p)$. In other words the orbit of x takes $m+n$ iterates from when it leaves the ball of radius ε around p, $B(p,\varepsilon)$ until it returns to $B(p,\varepsilon)$ for ever. So we can say that up to an ε-error x is a periodic point of period $m+n$.



The closures of sets of transverse homoclinic points of periodic points appear in many other problems in Dynamical System, for instance see [23], [36], and in many of these cases we cannot assume that they are hyperbolic. In contrast to the hyperbolic case, we know very little when a homoclinic closure is not hyperbolic. But from the ergodic theory point of view, at least on surfaces, they can be studied through the work of Pesin and Katok, previously discussed in Chapter 2.

In §1 we prove that the closure of the set of transverse homoclinic points of a periodic orbit $\theta(p)$, denoted by $C\ell H(\theta(p))$, has some topological dynamical properties similar to those of basic sets for Axiom A diffeomorphisms and discuss their role in ergodic theory of diffeomorphisms.

Sections 2 and 3 are devoted to establishing a relation between the growth rate of homoclinic orbits and topological entropy in the hyperbolic and general cases.

§1. Topological dynamics of homoclinic closures.

Let $f:M \to M$ be a C^1 diffeomorphism of a compact finite dimensional manifold M. If $p \in M$ is a hyperbolic periodic point of f, then a point $x \in W^u(p) \cap W^s(p)\backslash\{p\}$ is called a homoclinic point for p. If the intersection of $W^u(p)$ and $W^s(p)$ at x is transverse, i.e. $T_xM = T_xW^s(p) \oplus T_xW^u(p)$, the homoclinic point x is called transverse. Let us define

$$H(p) = \{x \in M : x \text{ is a transverse homoclinic point for } p\} ,$$

and if $\theta(p)$ denotes the orbit of p and n is its period,

$$H(\theta(p)) = \bigcup_{i=1}^{n-1} H(f^i(p)) .$$

We shall call $C\ell\, H(p)$ the <u>homoclinic closure</u> of p .

If p_1, p_2 are hyperbolic periodic points we say that p_1 is <u>h-related to</u> p_2 if $W^u(\theta(p_1))$ has a non-empty transverse intersection with $W^s(\theta(p_2))$ and $W^s(\theta(p_1))$ has a non-empty transverse intersection with $W^u(\theta(p_2))$. We write $p_1 \sim p_2$ if p_1 is h-related to p_2 , then $\sim$ defines a relation on the set of hyperbolic periodic points of f . This relation is clearly reflexive and symmetric. It follows from the λ-lemma [27], see [25] for details, that $\sim$ is transitive. Let us denote by $P(p)$ the equivalence class of hyperbolic periodic points h-related to p .

A homeomorphism $T:X \to X$ of a compact metric space X is called <u>topological transitive</u> if given any two non-emtpy open sets $U,V \subset X$ there exists $k \in \mathbb{Z}$ with

$$T^k U \cap V \neq \phi \quad .$$

And T is <u>topological mixing</u> if for any two open sets $U,V \subset X$ there exists $N > 0$ such that for all $k \geq N$

$$T^k U \cap V \neq \phi \ .$$

Proposition 5.1.1.

Let $f:M \to M$ *be as above*. *If* p *is a hyperbolic periodic point of* f *of period* n , *then*:

i) $C\ell H(\theta(p))$ *is a closed* f-*invariant set*.

ii) $C\ell H(\theta(p)) = C\ell P(p)$.

iii) $C\ell H(\theta(p)) = C\ell H(p) \cup \ldots \cup C\ H(f^{n-1}(p))$, *furthermore*,

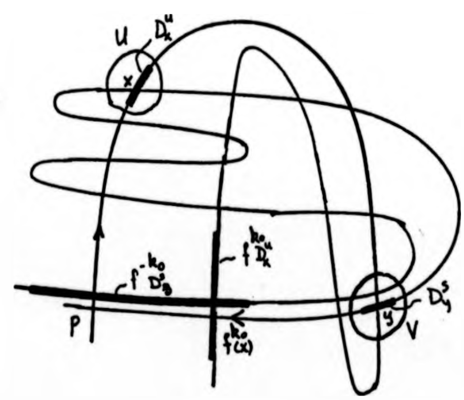
$f\ C\ell H(f^i(p)) = C\ell H(f^{i+1}(p))$.

iv) $f|C\ell H(\theta(p))$ *is topological transitive and* $f^n|C\ell H(p)$ *is topological mixing*.

Proof.

The proofs of i) and iii) are obvious, and ii) follows from Smale's homoclinic points theorem [23], [36]. To prove iv) is sufficient to show that if p is a fixed point then $f|C\ell H(p)$ is topological mixing. For this we shall consider $C\ell H(p)$ as a metric space with the induced metric and topology. Let U,V be any open sets in $C\ell H(p)$ and take $x \in U \cap H(p)$ and $y \in V \cap H(p)$. Now choose a small u-disk D_x^u on $W^u(p)$ containing x, similarly choose a small s-disk D_y^s on $W^s(p)$ containing y. Here $u = \dim W^u(p)$ and $s = \dim W^s(p)$. Suppose that $D_x^u \cap C\ell H(p) \subset U$ and $D_y^s \cap C\ell H(p) \subset V$.

By the λ-lemma [27] $\bigcup_{k=0}^{\infty} f^k D_x^u$ contains u-disks arbitrarily C^1 close to $W_{loc}^u(p)$, similarly $\bigcup_{k=0}^{\infty} f^{-k} D_y^s$ contains s-disks arbitrarily C^1 close to $W_{loc}^s(p)$. Therefore for some large positive integer k_0, $f^{k_0} D_x^u$ intersects $f^{-k_0} D_y^s$ transversally, and there since p is fixed for any $k \geq k_0$, $f^k D_x^u$ intersects $f^{-k_0} D_y^s$ transversally. And the proposition follows. □



The following statement shows how important the closures of homoclinic classes are to the study of ergodic theory of diffeomorphisms.

Proposition 5.1.2. [13]

Let $f:M \to M$ be a C^2 diffeomorphism of a compact manifold M. Suppose that μ is an ergodic f-invariant measure with nonzero exponents, then there exists a hyperbolic periodic point such that $\operatorname{supp}\mu \subset C\ell H(\theta(p))$.

Proof.

It follows from Theorem 4.2 of [13] that if $x_0 \in \Lambda^k_{\chi,\ell,\mu}$, where $\mu(\Lambda^k_{\chi,\ell,\mu}) > 0$, then for small $\delta > 0$

$$B(x_0,\delta) \cap \Lambda^k_{\chi,\ell,\mu} \subset C\ell H(\theta(p)) \text{ for some hyperbolic periodic point } p .$$

Since $C\ell H(\theta(p))$ is f-invariant and $\mu(B(x_0,\delta) \cap \Lambda^k_{\chi,\ell,\mu}) > 0$ it follows that $\sup p_\mu \subset C\ell H(\theta(p))$. $\square$

We shall use in this chapter a definition of topological entropy, also due to Bowen [7], for homeomorphisms of compact metric spaces.

Let $T:X \to X$ be a homeomorphism of a compact metric space X with metric d. A set $E \subset X$ is said to be (n,ε)-separated if for any $x,y \in E$ there exists $i \in [0,n)$ such that $d(T^i(x),T^i(y)) > \varepsilon$. If $s(n,\varepsilon)$ denotes the maximum cardinality of (n,ε)-separated sets, then

$$h(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon) .$$

Also let us recall that T , as above, is said to be expansive if there exists $\varepsilon > 0$ such that if $d(T^n(x),T^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$, then $x = y$.

§2. Counting homoclinic orbits of Axiom A basic sets.

We recall that if $f:M \to M$ is a C^1 diffeomorphism of a compact manifold M , f is said to satisfy Axiom A if

i) $\Omega(f)$ is hyperbolic,

ii) $\mathrm{C\ell Per}(f) = \Omega(f)$,

where $\mathrm{Per}(f) = \{x \in M \mid f^n(x) = x$ for some $n > 0\}$.

The Spectral Decomposition Theorem [6], [35] says that if f satisfies Axiom A , then

$$\Omega(f) = \Lambda_1 \cup \ldots \cup \Lambda_t ,$$

where each Λ_i , $1 \le i \le t$, is a closed f-invariant set, moreover $f|\Lambda_i$ is topological transitive and if $i \ne j$ then $\Lambda_i \cap \Lambda_j = \phi$. Bowen [6] noticed that each Λ_i could be decomposed as follows

$$\Lambda_i = \bigcup_{j=0}^{n-1} \Lambda_{i,j}$$

with $f\Lambda_{i,j} = \Lambda_{i,j+1} (\mathrm{mod}\ n)$ and $f^n|\Lambda_{i,j}$ is topological mixing for any $0 \leq j < n$. The sets $\Lambda_1,\ldots,\Lambda_t$ are called <u>basic sets</u> for f .

If Λ_i , a basic set, is not a single periodic orbit then for any periodic point $p \in \Lambda_i$, $\Lambda_i = \mathrm{C\ell H}(\theta(p))$, furthermore $h(\Lambda_i,f) > 0$ iff $H(\theta(p)) \neq \phi$. In [5], see also [6], Bowen proved that

$$h(\Lambda_i,f) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(f|\Lambda_i) \, ,$$

where $N_n(f|\Lambda_i) = \#\{x \in \Lambda_i : f^n(x) = x\}$.

A homoclinic point for a periodic point p resembles, in a way, the orbit of a periodic point of infinite period. So we shall define an order for homoclinic points, a sort of period, and prove a formula similar to Bowen's. In fact the method used in the proof is due to him, see [6].

Let p be a hyperbolic periodic point of f , and denote by $W^s_\varepsilon(p)$ the stable manifold of size ε , i.e.

$$\{x \in M : d(f^n(x),f^n(p)) < \varepsilon \text{ for all } n \geq 0\} \, ,$$

respectively denote by $W^u_\varepsilon(p)$ the unstable manifold of size ε , i.e. the set $\{x \in M : d(f^{-n}(x),f^{-n}(p)) < \varepsilon$ for all $n \geq 0\}$.

Now fix $\varepsilon > 0$ small enough. If $H(p) \neq \phi$ let us define the ε-order of $x \in H(p)$ as follows: let

$$\Theta^s(x,p,\varepsilon) = \min\{n \mid f^n(x) \in W^s_\varepsilon(p)\} \quad ,$$

$$\Theta^u(x,p,\varepsilon) = \min\{n \mid f^{-n}(x) \in W^u_\varepsilon(p)\} \quad ,$$

and

$$\Theta(x,p,\varepsilon) = \Theta^s(x,p,\varepsilon) + \Theta^u(x,p,\varepsilon) \ .$$

If $\Theta(x,p,\varepsilon) = n$, then x is said to be a homoclinic point of ε-order n . We could say in an experimentalist's language that "x is a periodic point of period n up to an ε-error".

Proposition 5.2.1.

i) For any $k \in \mathbb{Z}$, $\Theta(x,p,\varepsilon) = \Theta(f^k(x),f^k(p),\varepsilon)$.

ii) Suppose p has period n and let $\Theta_n(x,p,\varepsilon)$ denote the ε-order of x with respect to f^n , then

$$\Theta_n(x,p,\varepsilon) = \frac{1}{n}\,\Theta(x,p,\varepsilon) \ .$$

The proof of the above proposition is omitted because it is obvious. □

Now let

$$H(p,n,\varepsilon) = \{x \in H(p) : \Theta^s(x,p,\varepsilon) = n \ \text{ and } \ \Theta^u(x,p,\varepsilon) = 0\}$$

and let $h(p,n,\varepsilon) = \# H(p,n,\varepsilon)$.

Theorem 5.2.2.

Let $f:M \to M$ be a C^1 diffeomorphism of a compact manifold M satisfying Axiom A, then if $h(f) > 0$ there exists a periodic point p such that for small $\varepsilon > 0$

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon) .$$

Proof.

By the Spectral Decomposition Theorem there exists a basic set Λ such that $h(f) = h(f|\Lambda)$. It is clear that for any periodic point $p \in \Lambda$, then $C\ell H(p) = \Lambda$. So take $p \in \Lambda$ and assume that p is fixed, otherwise take a power of f that fixes p . This implies that Λ is topological mixing, by Proposition 5.1.1.

The Stable Manifold Theorem of Hirsch and Pugh [12] implies that $f|\Lambda$ is expansive, so let $\delta > 0$ as in the definition of expansiveness and choose $0 < \varepsilon < \delta/4$. Now if $x,y \in H(p,n,\varepsilon)$ and $d(f^k(x),f^k(y)) < \varepsilon$ for any $0 \le k < n$, then the definition of ε-order implies that $d(f^k(x),f^k(y)) < \varepsilon$ for all $k \in \mathbb{Z}$ and by expansiveness $x = y$. Hence if $x \ne y$ then there exists $k \in [0,n)$ such that $d(f^k(x),f^k(y)) > \varepsilon$, and $h(p,n,\varepsilon) \le s(n,\varepsilon)$. Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon) \le h(f) .$$

To complete the proof of the theorem we shall use Bowen's method of pseudo-orbits [8]. A sequence $\underline{x} = \{x_i\}_{i=a}^{b}$ (with $a = -\infty$ or $b = -\infty$ permitted) of points in M is an α-<u>pseudo-orbit</u>, for some $\alpha > 0$, if $d(f(x_i),x_{i+1}) < \alpha$ for all $i \in [a,b)$. A point $x \in M$ is said to β-<u>shadow</u> $\underline{x}$, for $\beta > 0$, if

$$d(f^i(x),x_i) \le \beta \quad \text{for all } i \in [a,b] .$$

<u>Lemma 5.2.3.</u> (Shadowing Lemma) [8]

<u>Let</u> Λ <u>be a basic set for</u> f <u>satisfying Axiom</u> A, <u>then for every</u> $\beta > 0$ <u>there exists an</u> $\alpha > 0$ <u>so that every</u> α-<u>pseudo-orbit</u> $\{x_i\}_{i=a}^{b}$ <u>in</u> Λ <u>is</u> β-<u>shadowed by a point</u> $x \in \Lambda$. □

Continuing the proof of Theorem 5.2.1 we shall shadow some sets of separated points by homoclinic points. But, first observe that if $x \in H(p,n,\varepsilon)$, then $x \in C\ell(W_\varepsilon^u(p)\backslash f^{-1}W_\varepsilon^u(p))$ and $f^n(x) \in C\ell(W_\varepsilon^s(p)\backslash fW_\varepsilon^s(p))$.

Let $0 < \beta < \varepsilon/8$ and α be as in the Shadowing Lemma. By compactness of Λ we can take a finite cover $\mathcal{U}$ of Λ by α-balls. Since $f|\Lambda$ is topological mixing there exists $N = N(\mathcal{U}) > 0$ such that for any two balls $B_i, B_j \in \mathcal{U}$

$$f^{-n}B_i \cap B_j \cap \Lambda \neq \phi \qquad \forall\, n \ge N .$$

Choose $w_1 \in (W_\varepsilon^s(p)\backslash fW_\varepsilon^s(p)) \cap H(p)$ and $w_2 \in (W_\varepsilon^u(p)\backslash f_\varepsilon^{-1}W^u(p)) \cap H(p)$, and assume that $B(w_1,\alpha)$ and $B(w_2,\alpha)$ belong to the cover $\mathcal{U}$. Now let E be an (n,ε)-separated set in Λ , for $x \in E$ take $u(x) \in f^{-N}B(x,\alpha) \cap B(w_2,\alpha) \cap \Lambda$ and $r(x) \in f^{N}B(f^{n}(x),\alpha) \cap B(w_1,\alpha) \cap \Lambda$, (we shall assume that f preserves orientation) and consider the following sequence

$$\ldots f^{-1}(w_2), u(x), f(u(x)), \ldots, f^{N-1}(u(x)), x, \ldots, f^{n-1}x, f^{-N}(v(x)), \ldots, f^{-1}v(x), w_1, f(w_1), \ldots$$

by construction the above sequence is an α-pseudo-orbit contained in Λ , and by the Shadowing Lemma there exists $z(x) \in \Lambda$ that β-shadows it.

Now we have to show that $z(x) \in H(p)$. If $d(x,z(x)) < \beta$ then $d(f^{k}(w_1), f^{k+n+N}(z(x))) < \beta$ for all $k \geq 0$, which implies that $f^{k}(z(x)) \to p$ as $k \to \infty$. Similarly $f^{-k}(z(x)) \to p$ as $k \to \infty$, and therefore $z(x) \in H(p)$, furthermore if α is chosen small enough the hyperbolic structure around p implies that $\theta(z(x),p,\varepsilon) = 2N+n$.

If $x,y \in E$, then $z(x) \neq z(y)$ because

$$0 < \varepsilon-2\beta < \varepsilon-d(f^{k}(x),f^{k}(z(x)))-d(f^{k}(y),f^{k}(z(y))) \leq d(f^{k}(z(x)),f^{k}(z(y)))$$

for some $k \in [0,n)$, since E is (n,ε)-separated. Therefore $s(n,\varepsilon) \leq h(p,n+2N,\varepsilon)$ for all $n \geq 0$ and N is independent of n and ε. Hence by expansiveness

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon) .$$

If p was not a fixed point and the above proof was carried out with a power of f, then using Proposition 5.2.1 ii) the result follows. □

Corollary 5.2.4.

If Λ *contains a fixed point* p, *then for small* $\varepsilon > 0$

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon) \ .$$

Proof.

By the local product structure of Λ, see [8] for definition, for small $\delta > 0$ there exists $\varepsilon > 0$ such that for any $x,y \in \Lambda$ $W^s_\delta(x)$ intersects $W^u_\delta(y)$ at a single point $[x,y]$ if $d(x,y) < \varepsilon$. Consider the set

$$A_n = \{x \in H(p,n,\varepsilon) \mid f^n W^s_\delta(x) \subset W^s_\varepsilon(p) \setminus fW^s_\varepsilon(p) \quad \text{and}$$
$$f^{-n}W^u_\delta(f^n(x)) \subset W^u_\varepsilon(p) \setminus f^{-1}W^u(p)\} \ .$$

It is not difficult to see that $h(p,n,\varepsilon) \le \# A_n + 4$.

If $x \in A_m$ and $y \in A_n$, $z = [x, f^n(y)] \in H(p)$ and $\Theta(p,z,\varepsilon) = n+m$ and if $\varepsilon << \delta$, $z \in A_{m+n}$. Hence $\# A_m \# A_n \le \# A_{m+n}$, and by a standard argument it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log \# A_n = \sup_n \frac{1}{n} \log \# A_n \ .$$

Therefore since $\# A_n \leq h(p,n,\varepsilon) \leq \# A_n + 4$,

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon) \ . \qquad \square$$

Remark 5.2.5.

Let us suppose that the diffeomorphism $f:M \to M$ does not necessarily satisfy Axiom A.

i) By the proof of Theorem 5.2.2 it follows that if $C\ell H(\theta(p))$ is hyperbolic then for small $\varepsilon > 0$

$$h(C\ell H(\theta(p)),f) = \limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon) \ .$$

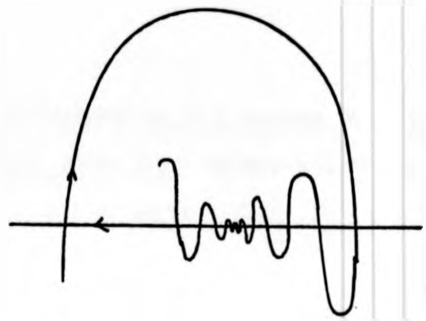
ii) If $C\ell H(\theta(p))$ is an n-folds horseshoe of period k , then

$$h(C\ell H(\theta(p)),f) = \frac{1}{m} \log h(p,m,\varepsilon)$$

for any small $\varepsilon > 0$ and $m > 0$ a large multiple of k . $\square$

§3. Entropy of non-uniform hyperbolic closures.

In this section we shall study homoclinic closures for C^2 diffeomorphisms of surfaces. The method used in §2 of counting homoclinic orbits no longer works if $C\ell H(\theta(p))$ is not hyperbolic. We cannot even guarantee that $H(p,n,\varepsilon)$ is finite, for instance if the unstable manifold of p locally looks like $x \to x^5$ sen $1/x$ over an interval of $W^s(p)$, as in the picture below.



Using Pesin's theory of non-uniform hyperbolicity we can reformulate the way of counting the different homoclinic orbits. Suppose that $f:M \to M$ is a C^2 diffeomorphism of a surface M. If p is a hyperbolic periodic point for f of least period k, let $|\lambda_1| < 1 < |\lambda_2|$ denote the moduli of the eigenvalues of D_pf^k (otherwise if p is not of saddle type $H(p) = \phi$). Then for $x \in H(p)$ we have that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log||D_xf^nv|| = \frac{1}{k} \log|\lambda_2| \quad \text{for } v \in T_xW^u(p) ,$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log||D_xf^nv|| = \frac{1}{k} \log|\lambda_1| \quad \text{for } v \in T_xW^s(p) .$$

Hence if $\chi(p) = \max\{-\frac{1}{k}\log|\lambda_1| , \frac{1}{k}\log|\lambda_2|\}$, then for each $x \in H(\theta(p))$ there exists $\ell = \ell(x) \geq 1$ so that for any $0 < \chi \leq \chi(p)$, $x \in \Lambda_{\chi,\ell}$.

For small $\varepsilon > 0$, $0 < \chi \leq \chi(p)$, $\ell \geq 1$ and $n > 0$ let

$$H(p,n,\varepsilon,\chi,\ell) = \{x \in H(p,n,\varepsilon) \cap \Lambda_{\chi,\ell} \mid f^n(x) \in \Lambda_{\chi,\ell}\} ,$$

and write $h(p,n,\varepsilon,\chi,\ell)$ for $\#H(p,n,\varepsilon,\chi,\ell)$.

Proposition 5.3.1.

Let $f:M \to M$ be a C^2 diffeomorphism of a surface M. Suppose that p is a hyperbolic periodic point with $h(C\ell H(\theta(p)),f) > 0$. Then for any $0 < X \le X(p)$, $\varepsilon > 0$ small and $\ell \ge 1$

$$\limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon,X,\ell) \le h(C\ell H(\theta(p)),f) .$$

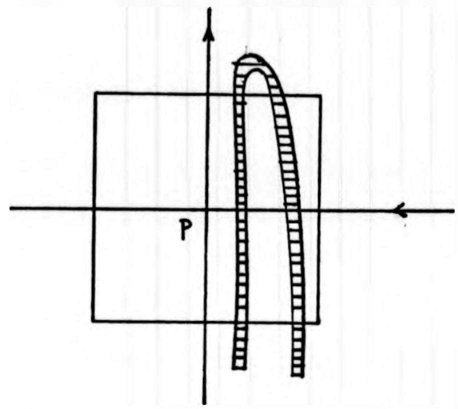
Proof.

First let us observe that if k is the least period of p, it is not difficult to verify that if $x \in H_k(p,n,\varepsilon,X,\ell)$ then $x \in H(p,nk,X/k,C^k\ell \exp X(1-10^{-3})^k)$, where $C = \sup_{x\in M}||D_xf||$. Here $H_k(p,n,\varepsilon,X,\ell) = \{x \in H_k(p,n,\varepsilon) \cap \Lambda_{X,\ell} \mid f^n(x) \in \Lambda_{X,\ell}\}$. Thus we may assume that p is fixed.

The idea of the proof is to construct a horseshoe Λ of $h(p,n,\varepsilon,X,\ell)$ folds of period n, with $\Lambda \subset C\ell H(p)$. For this we shall use the method introduced in Chapter 2 of constructing horseshoes.

As in §1 of Chapter 3, let $0<h<\frac{1}{4}$ be so small that for any $x \in \hat{C}(p,h) \cap \Lambda_{X,\ell}$, $W^s_{loc}(x)$ is an admissible $(s,1)$-manifold near p and $W^u_{loc}(f^n(x))$ is an admissible $(u,1)$-manifold near p. Let $\rho = \psi(X,h,\ell)$ as in Proposition 2.1.7, then using the same arguments as in the outline of the proof of Theorem 2.1.8 it follows that

$\hat{C}(p,h)$ is a (ρ,λ)-rectangle cover for $B(p,\rho) \cap \Lambda_{X,\ell}$, consisting of one rectangle.



By ii) of the definition of a (ρ,h)-rectangle cover, then $C(f^n(x), f^n\hat{C}(p,h) \cap \hat{C}(p,h))$ is a u-rectangle in $\hat{C}(p,h)$ and $f^{-n}C(f^n(x), f^n\hat{C}(p,h) \cap \hat{C}(p,h))$ is a s-rectangle in $\hat{C}(p,h)$.

For $x \in H(p,n,\rho,X,\ell)$ let $D^s_x = W^s_{loc}(x) \cap \hat{C}(p,h)$ and $D^u_{f^n(x)} = W^u_{loc}(f^n(x)) \cap \hat{C}(p,h)$. We may assume, by a similar argument to the one used in Corollary 5.2.4, that $f^n D^s_x \subset W^s_\rho(p)$ and $f^{-n}D^u_{f^n(x)} \subset W^u_\rho(p)$. Hence if $x \neq y \in H(p,n,\rho,X,\ell)$ then $D^s_x \cap D^s_y = \phi$ and $D^u_{f^n(x)} \cap D^u_{f^n(x)} = \phi$. Therefore

$$C(f^n(x), f^n\hat{C}(p,h) \cap \hat{C}(p,h)) \cap C(f^n(y), f^n\hat{C}(p,h) \cap \hat{C}(p,h)) = \phi .$$

From this it is not difficult to verify that we have a $h(p,n,\rho,X,\ell)$—folds horseshoe Λ of period n contained in $\hat{C}(p,h)$. Furthermore $\Lambda \subset C\ell H(p)$. From this it follows that for small $\rho > 0$

$$\frac{1}{n} \log h(p,n,\rho,X,\ell) \le h(C\ell H(p),f) . \qquad \square$$

<u>Note</u>.

The above proof should work in higher dimensional manifolds with complications only in the notation. □

We shall say that $C\ell H(\theta(p))$ is <u>isolated</u> if there exists an open set U such that $C\ell H(\theta(p)) \subset U$ and $U \cap \Omega(f) = C\ell H(\theta(p))$.

<u>Lemma 5.3.2</u>.

<u>If</u> $C\ell H(\theta(p))$ <u>is isolated then</u>

$$h(C\ell H(\theta(p)),f) = \sup\{h(\Lambda,f) : \Lambda \subset C\ell H((\theta(p)) \text{ is a horseshoe}\} .$$

<u>Proof</u>.

This follows from Corollary 2.3.4. □

<u>Theorem 5.3.3</u>.

<u>Let</u> $f:M \to M$ <u>be a</u> C^2 <u>diffeomorphism of a surface</u> M <u>and let</u> p <u>be a hyperbolic periodic point of</u> f. <u>If</u> $H(\theta(p)) \ne \phi$ <u>and</u> $C\ell H(\theta(p))$ <u>is isolated, then</u>

$$h(C\ell H(\theta(p)),f) = \sup_{0<X\le X(p)} \sup_{\ell\ge 1} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon,X,\ell) \ .$$

Proof.

Without loss of generality let us suppose that p is a fixed point. By Proposition 5.3.1 it is sufficient to prove that for any $\beta > 0$ there exist a small $\varepsilon > 0$, $X > 0$ and $\ell \ge 1$ so that

$$h(C\ell H(p),f) \le \limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon,X,\ell) + \beta \ .$$

The idea of the proof is to use Lemma 5.3.2 to find a horseshoe $\Lambda \subset C\ell H(p)$ with entropy as large as we want. Then choose a periodic point $p' \in \Lambda$, since $C\ell\,(p)$ is isolated $p \sim p'$. Using the fact that $p \sim p'$ and the hyperbolicity of Λ we construct a transverse homoclinic point $x = x(y)$ of p for each homoclinic point $y \in H(\theta(p')) \cap \Lambda$. The hyperbolicity along the orbit of x will be determined basically by that for y and by D_pf. The sets $H(p',n,\rho) \cap \Lambda$ will play the role of the (n,ε)-separated sets of the proof of Theorem 5.2.2 and the points $x = x(y)$, for $y \in H(p',n,\rho) \cap \Lambda$, will shadow the orbit of y up to n iterates.

So by Lemma 5.3.2 for $\beta > 0$ fixed, choose a horseshoe $\Lambda \subset C\ell H(p)$ such that

$$h(C\ell H(p),f) \le h(\Lambda,f) + \beta \ .$$

We may assume that Λ contains a fixed point p' and that there exists $\chi_1 > 0$ such that for $y \in \Lambda$, $n \geq 0$

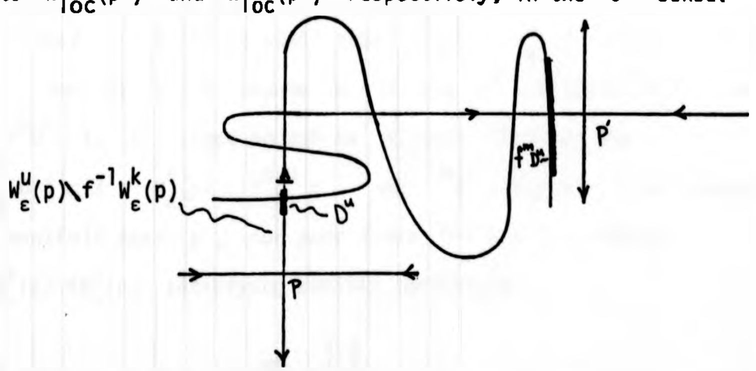
$$||D_y f^n v|| \leq \exp -\chi_1 n ||v|| \qquad \text{for } v \in E^s_y$$

and

$$||D_y f^n v|| \geq \exp \chi_1 n ||v|| \qquad \text{for } v \in E^u_y .$$

This is possible by taking a large power of f.

Let $\varepsilon > 0$ be so small that if $x \in H(p,n,\varepsilon)$, for $n \geq 0$, there exists $\ell \geq 1$ such that $x \in H(p,n,\varepsilon,\chi(p),\ell)$. Since $p \sim p'$, the λ-lemma [27] enables us to choose $m > 0$, $D^u \subset W^u_\varepsilon(p)\backslash f^{-1}W^u_\varepsilon(p)$ and $D^s \subset W^s_\varepsilon(p)\backslash fW^s_\varepsilon(p)$ such that $f^m D^u$ and $f^{-m}D^s$ are as close as we wish to $W^u_{loc}(p')$ and $W^s_{loc}(p')$ respectively, in the C^1 sense.



Let us recall some facts, previously used in this thesis, that would tell us how close $f^m D^u$ and $f^{-m}D^s$ should be to $W^u_{loc}(p')$ and $W^s_{loc}(p')$

respectively, to allow us to construct the new homoclinic points.

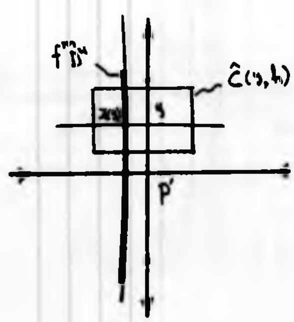
As in Lemma 4.2.3 (since f is C^2) for a fixed $\alpha > 0$ choose $a > 0$ and $c > 0$ such that if $y \in \Lambda$, $z \in M$ and $d(z,y) < a$, then if W is a C^1 submanifold containing z and $\exp_y^{-1} W$ is an (E_y^s, E_y^u)-graph with dispersion $\le c$,

$$\big|\, ||D_y f | E_y^u|| - ||D_z f | T_z W||\,\big| < \alpha \quad .$$

And if $\exp_y^{-1} W$ is an (E_y^u, E_y^s)-graph with dispersion $\le c$,

$$\big|\, ||D_y f | E_y^s|| - ||D_z f | T_z W||\,\big| < \alpha \quad .$$

The uniform hyperbolicity of Λ implies that for any $0 < h \le 1$ the size of the Lyapunov chart $C(y,h)$, $y \in \Lambda$, is constant on Λ. Suppose that $a > 0$ is so small that $W_a^s(y)$, $W_a^u(y) \subset C(y,1)$ for any $y \in \Lambda$. Now for $\rho > 0$ choose $m > 0$ and $D^u \subset W_\varepsilon^u(p) \setminus f^{-1} W_\varepsilon^u(p)$ so that $f^m D^u$ is C^1 close enough to $W_{2\rho}^u(p')$ that for any $y \in W_\rho^u(p') \cap \Lambda$: $W_a^s(y) \cap f^m D^u \ne \phi$ and $f^m D^u \cap \hat{C}(y,h)$ is an admissible (u,h)-manifold near y, for some fixed $0 < h \le 1$. Choose $D^s \subset W_\varepsilon^s(p) \setminus f W_\varepsilon^s(p)$ satisfying similar conditions.



Fix $\rho > 0$ such that $2a < \rho$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \#(H(p',n,\rho) \cap \Lambda) = h(\Lambda,f) .$$

Thus if $y \in H(p',n,\rho) \cap \Lambda$, consider $z = z(y) \in W_a^s(y) \cap f^m D^u$ and $V_0(y) = f^m D^u \cap \hat{C}(y,h)$. As in Proposition 2.1.7, define

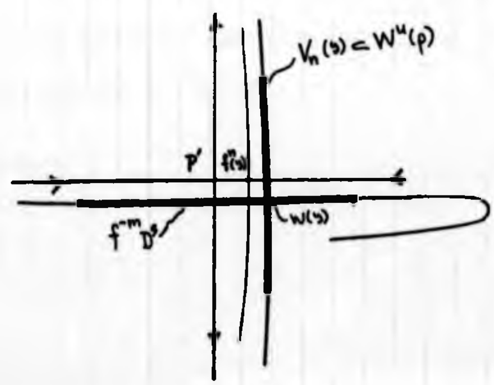
$$V_1(y) = f(V_0(y)) \cap \hat{C}(f(y),h) ,$$

$$V_2(y) = f(V_1(y)) \cap \hat{C}(f^2(y),h) ,$$

$$\vdots$$

$$V_n(y) = f(V_{n-1}(y)) \cap \hat{C}(f^n(y),h) .$$

For each $0 \le k \le n$, $V_k(y)$ is an admissible (u,h)-manifold near $f^k(y)$. Therefore $V_n(y)$ intersects $f^{-m}D^s$ transversely at a point $w = w(y) \in H(p)$, so let $x = x(y) = f^{-n}w$. For $h > 0$ sufficiently small $d(f^k(y),f^k(x(y))) < a$ for all $0 \le k \le n$.



By construction $\theta(x,p,\varepsilon) = n+2m$, and m is clearly independent of n . So it remains to find $X > 0$ and $\ell \geq 1$ such that for any $x = x(y) \in H(p)$ constructed as above for $y \in H(p',n,\varepsilon) \cap \Lambda$, $f^{-m}(x) \in \Lambda_{X,\ell}$ and $f^{m+n}(x) \in \Lambda_{X,\ell}$. This will imply that

$$\#(H(p',n,\rho) \cap \Lambda) \leq h(p,n+2m,\varepsilon,X,\ell) ,$$

from which the Theorem follows.

For $x \in H(p)$ let $E^u_x = T_x W^u(p)$ and $E^s_x = T_x W^s(p)$, by transversality $T_x M = E^s_x \oplus E^u_x$. Recall that $\gamma(x)$ denotes the angle between the subspaces E^s_x and E^u_x . By the proof of the λ- lemma [27], if $\varepsilon > 0$ is sufficiently small, if $x \in H(p) \cap W^s_\varepsilon(p)$ and $\gamma(x) \geq t$ for some $t > 0$, then $\gamma(f^k(x)) \geq t$ for all $k \geq 0$. Similarly if $x \in H(p) \cap W^u_\varepsilon(p)$ and $\gamma(x) \geq t$, then $\gamma(f^{-k}(x)) \geq t$ for all $k \geq 0$.

Now for $x = x(y) \in H(p)$, $y \in H(p',n,\rho) \cap \Lambda$, the angle $\gamma(f^k(x))$ is bounded away from zero, say greater than or equal to some $r > 0$, independent of n , for all $0 \leq k \leq n$. This is so because $f^k(x)$ is the intersection of two admissible manifolds near $f^k(y)$. By linear algebra we have that for any $0 < k \leq m$

$$\operatorname{sen} \gamma(f^{-k}(x)) \geq (\operatorname{sen} r)(\inf_{w \in M}|\det D_w f|)^m (\sup_{w \in M}||D_w f||)^{-2m} ,$$

and

$$\text{sen}\ \gamma(f^{n+k}(x)) \ge (\text{sen}\ r)(\inf_{w\in M}|\det D_w f|)^m (\sup_{w\in M}\|D_w f\|)^{-2m} .$$

Since $f^{-m}(x) \in W^u_\varepsilon(p)$ and $f^{m+n}(x) \in W^s_\varepsilon(p)$, there exists $\ell_1 > 1$ such that $\gamma(f^i(x)) \ge \ell_1^{-1}$ for all $i \in \mathbb{Z}$.

Now we shall look at how the derivative acts on E^s_x , for $x = x(y)$ and $y \in H(p',n,\rho) \cap \Lambda$. Take $v \in E^s_x$, since $d(f^k(x),f^k(y)) < a$ for $0 \le k < n$, then it follows that if $\alpha \le \exp - \chi_1(\exp 10^{-3}-1)$

$$\begin{aligned}
\|D_x f^k v\| &\le \Big(\prod_{i=0}^{k-1} \|D_{f^i(x)} f \mid E^s_{f^i(x)}\|\Big)\|v\| \\
&\le \Big(\prod_{i=0}^{k-1} (\|D_{f^i(y)} f \mid E^s_{f^i(y)}\| + \alpha)\Big)\|v\| \\
&\le \exp-(\chi_1-10^{-3})k\|v\| .
\end{aligned}$$

If $\varepsilon > 0$ is small and $k \ge 0$, for $v \in E^s_{f^{m+n}(x)}$

$$\|D_{f^{m+n}(x)} f^k v\| \le \exp-(\chi(p)-10^{-3})k\|v\|$$

and for $v \in E^s_{f^{-m}(x)}$

$$\|D_{f^{-m}(x)} f^{-k} v\| \ge \exp-(\chi(p)-10^{-3})k\|v\| .$$

And for $0 < k \leq m$, if $C = \sup_{w\in M} ||D_w f||$, then

$$||D_{f^n(x)} f^k v|| \leq C^k ||v||$$

and

$$||D_x f^{-k} v|| \geq C^{-k} ||v|| \quad .$$

Similar inequalities hold for $v \in E^u_x$. Now let us set $X = \min\{X_1, X(p)\}$ and $\ell = \max\{\ell_1, (C \exp(X-10^{-3}))^{2m}\}$. Using the above inequalities it is straightforward to check that if $y \in H(p',n,\rho) \cap \Lambda$ for any $n > 0$, then $x = x(y) \in \Lambda_{X,\ell}$ and so do $f^{-m}(x)$ and $f^{m+n}(x)$. Since $\theta^u(f^{-m}(x),p,\varepsilon) = 0$ and $\theta^s(f^{-m}(x),p,\varepsilon) = 2m+n$, $f^{-m}(x) \in H(p,n+2m,\varepsilon,X,\ell)$ and it remains to prove that

$$\#(H(p',n,\rho) \cap \Lambda) \leq h(p,n+2m,\varepsilon,X,\ell) \ .$$

If $y_1,y_2 \in H(p',\rho,n) \cap \Lambda$, then for some $0 \leq k < n$ $d(f^k(y_1),f^k(y_2)) > \rho$ (by expansiveness of $f|\Lambda$) , so

$$\begin{aligned}\rho \leq d(f^k(y_1),f^k(y_2)) &\leq d(f^k(y_1),f^k(x(y_1)))+d(f^k(x(y_1)),f^k(x(y_2))) \\ &\quad + d(f^k(x(y_2)),f^k(y_2)) \\ &\leq 2a + d(f^k(x(y_1)),f^k(x(y_2))) \ .\end{aligned}$$

Hence since $2a < \rho$, it follows that $x(y_1) \neq x(y_2)$.

Now by taking limits where $n \to \infty$, we have

$$h(\Lambda,f) = \lim \frac{1}{n} \log \#(H(p',n,\rho) \cap \Lambda) \leq \limsup_{n\to\infty} \frac{1}{n} \log h(p,n,\varepsilon,X,\ell)$$

from where the theorem follows. $\square$

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